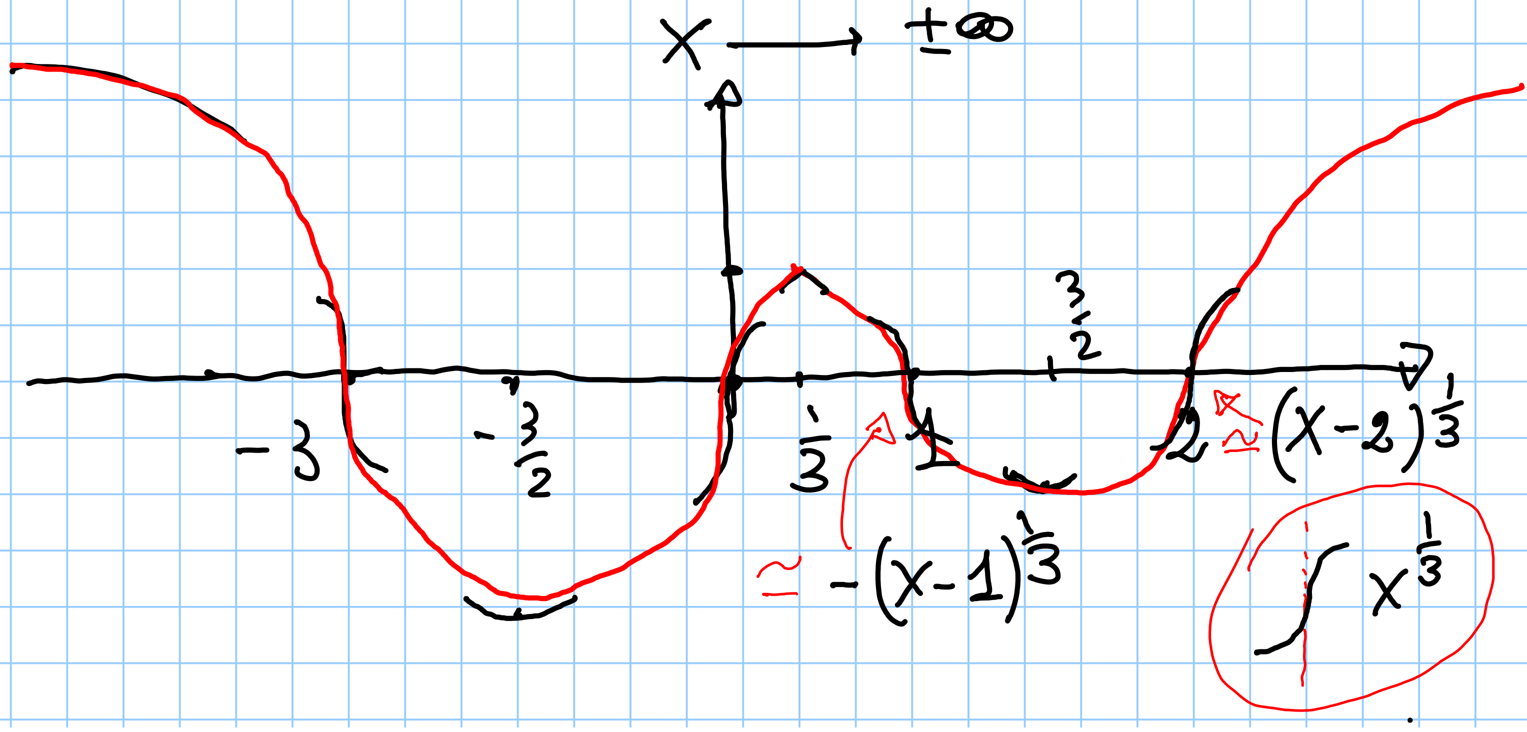
$$f(x) = (x^2 + 1 - |3x-1|)^{\frac{1}{3}}$$

$$\mathrm{dom}(f) = \mathbb{R}$$

$$\lim_{x \to \pm\infty} f(x) = +\infty \quad ; \quad f(x) = x^{2/3}(1 + o(1)) \quad x \to \pm\infty$$



$$f(x) = \begin{cases} \left(x^2+1-(3x-1)\right)^{\frac{1}{3}}, & x \geq \frac{1}{3} \\ \left(x^2+1+(3x-1)\right)^{\frac{1}{3}}, & x < \frac{1}{3} \end{cases}$$

$$= \begin{cases} \left(x^2-3x+2\right)^{1/3} & x \geq 1/3 \\ \left(x^2+3x\right)^{\frac{1}{3}}, & x < \frac{1}{3} \end{cases}$$

$$f'(x) = \begin{cases} \frac{1}{3}\,\frac{2x-3}{(x^2-3x+2)^{2/3}}, & x > \frac{1}{3} \\ & x \neq 1,2 \\ \frac{1}{3}\,\frac{2x+3}{(x^2+3x)^{2/3}}, & x < \frac{1}{3} \\ & x \neq 0,-3 \end{cases}$$

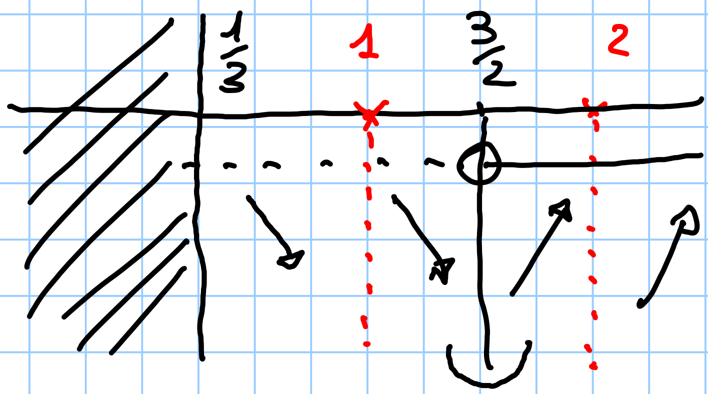
$$\mathrm{dom}(f') = \mathrm{dom}(f) \setminus \left\{\frac{1}{3}, -3, 0, 1, 2\right\}$$

$$\frac{1}{3}\,\frac{2x-3}{\underbrace{(x^2-3x+2)^{\frac{2}{3}}}_{\geq 0}} \geq 0 \qquad x \geq \frac{3}{2}, \quad \boxed{x > \frac{1}{3}}$$

$$x \neq 1, 2$$

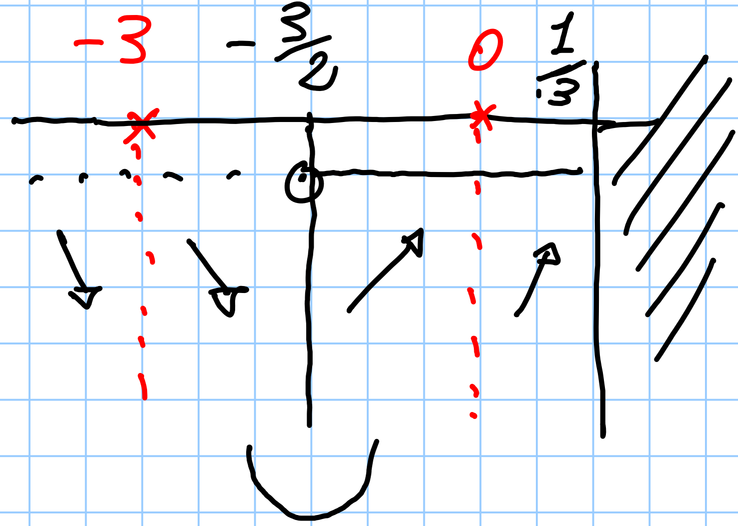
$x_0 = \frac{3}{2}$ MINIMO LOCALE

$$f\left(\frac{3}{2}\right) = \left(-\frac{1}{4}\right)^{\frac{1}{3}}$$



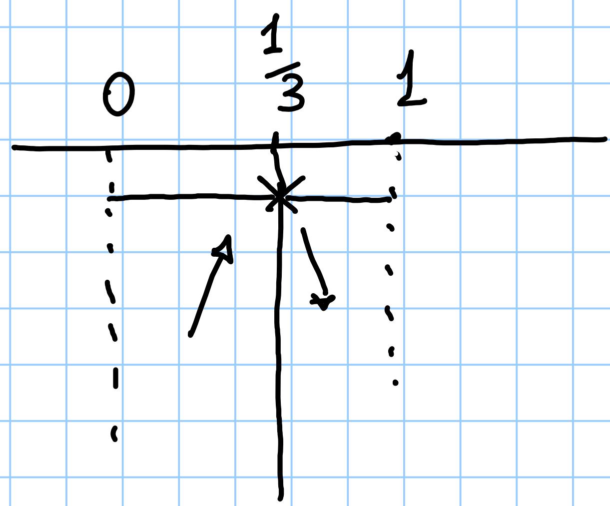
$$\frac{2x+3}{(x^2+3x)^{\frac{2}{3}}} \geq 0 \iff x \geq -\frac{3}{2}, \quad \boxed{x < \frac{1}{3}}$$

$$x \neq -3, 0$$



$x_0 = -\frac{3}{2}$ MINIMO LOCALE

$$f\left(-\frac{3}{2}\right) = \left(-\frac{9}{4}\right)^{\frac{1}{3}}$$



MASSIMO LOCALE

$x_0 = \frac{1}{3}$

$$f\left(\frac{1}{3}\right) = \left(\frac{10}{9}\right)^{\frac{1}{3}}$$

$$\lim_{x \to \left(\frac{1}{3}\right)^+} f'(x) = \frac{1}{3} \frac{\left(\frac{2}{3} - 3\right)}{\left(\frac{10}{9}\right)^{\frac{2}{3}}} < 0$$

$$\lim_{x \to \left(\frac{1}{3}\right)^-} f'(x) = \frac{1}{3} \frac{\frac{2}{3} + 3}{\left(\frac{10}{9}\right)^{\frac{2}{3}}} > 0$$

$x_0 = \frac{1}{3}$ PUNTO ANGOLOSO

$$\lim_{x \to 1} f'(x) = \lim_{x \to 1} \frac{2x-3}{3(x^2-3x+2)^{\frac{2}{3}}} = \frac{-1}{3 \cdot 0^+} = -\infty$$

$$\lim_{x \to 2} f'(x) = \lim_{x \to 2} \frac{2x-3}{3(x^2-3x+2)^{2/3}} = \frac{1}{3 \cdot 0^+} = +\infty$$

$x_0 = 1$ e $x_0 = 2$ PUNTI di TANGENZA VERTICALE

$$\boxed{x_0 = 0, \; x_0 = -3 \quad \text{PUNTI di TANGENZA VERTICALE}}$$

EX

$$\lim_{x \to +\infty} \sin(x) \ \nexists$$

TROVIAMO a_n e b_n:

$$a_n \to +\infty, \ \sin(a_n) = 0$$

$$b_n \to +\infty, \ \sin(b_n) = 1$$

$$\lim_n \sin(a_n) = 0 \neq 1 = \lim_n \sin(b_n)$$

IL LIMITE $\nexists$

$a_n = 2\pi n$, $\sin(a_n) = 0 \ \forall n$

$b_n = 2\pi n + \frac{\pi}{2}$, $\sin(b_n) = 1 \ \forall n$

Più in generale se $y \in [-1, 1]$

e $c_n = 2\pi n + \arcsin(y) \Rightarrow$

$$\sin(c_n) = y \quad \forall n \in \mathbb{N}$$

ALCUNI ESEMPI

$$f(x) \longrightarrow 0^+, \quad x \longrightarrow +\infty$$

non implica che f sia definitivamente $\downarrow$

COSTRUIAMO

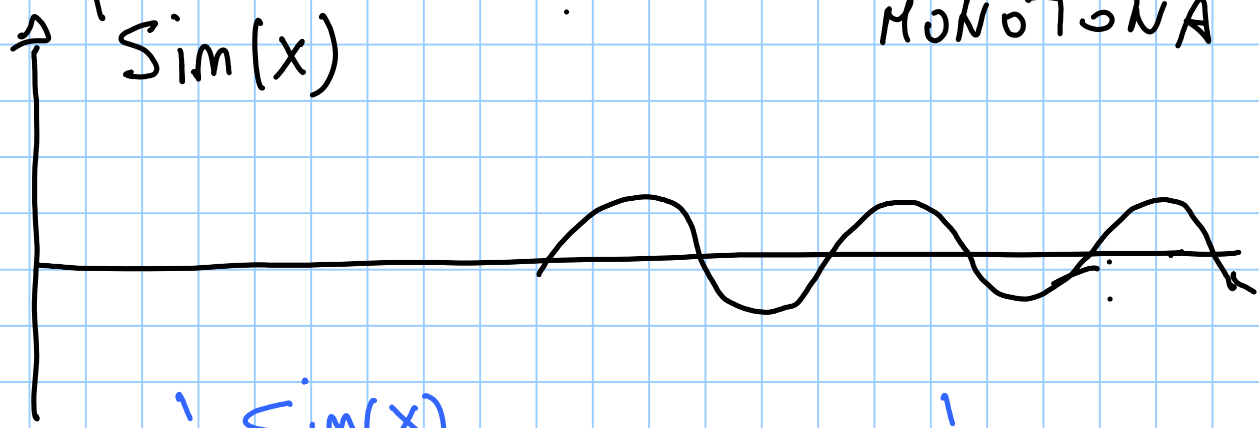
$f : [1, +\infty) \longrightarrow \mathbb{R}$

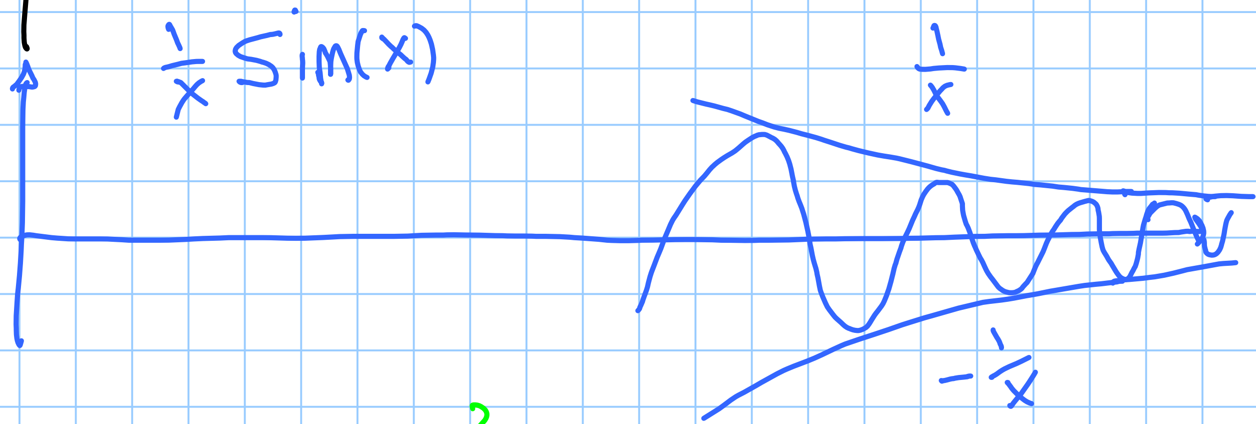
f derivabile in $[1, +\infty)$

$$\lim_{x \to +\infty} f(x) = 0^+$$

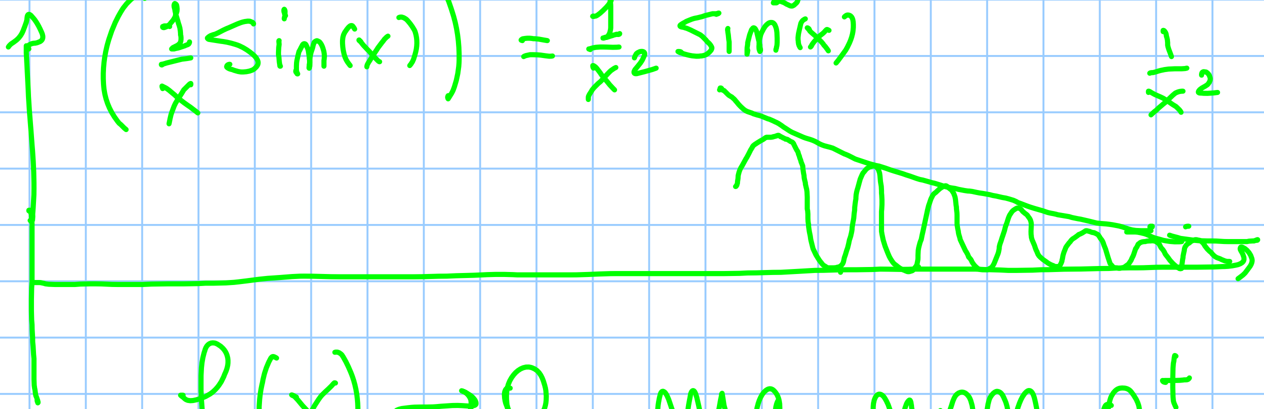
f non DEFINITIVAMENTE $\downarrow$

- $f(x) \longrightarrow 0^+ \qquad x \longrightarrow +\infty$
- f NON DEFINITIVAMENTE MONOTONA



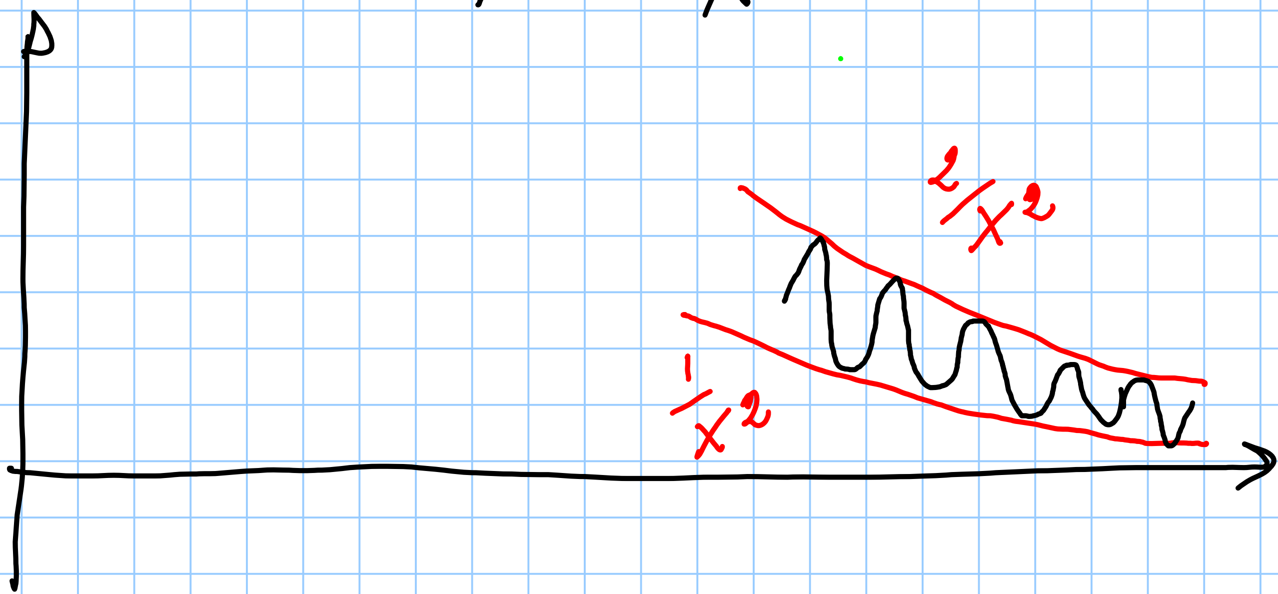


$$\left(\frac{1}{x}\sin(x)\right)^2 = \frac{1}{x^2}\sin^2(x)$$



$f(x) \to 0$ ma non 0^+

$$f(x) = \frac{1}{x^2} + \frac{1}{x^2}\sin^2(x)$$



$$\frac{1}{x^2} \leq f(x) \leq \frac{2}{x^2}$$

COSTRUIAMO

$f : [1, +\infty) \longrightarrow \mathbb{R}$

$$\lim_{x \to +\infty} f(x) = +\infty$$

$f'(x) > 0 \quad \forall x \in [1, +\infty)$

$f'(x)$ non DEFINITIVAMENTE MONOTONA

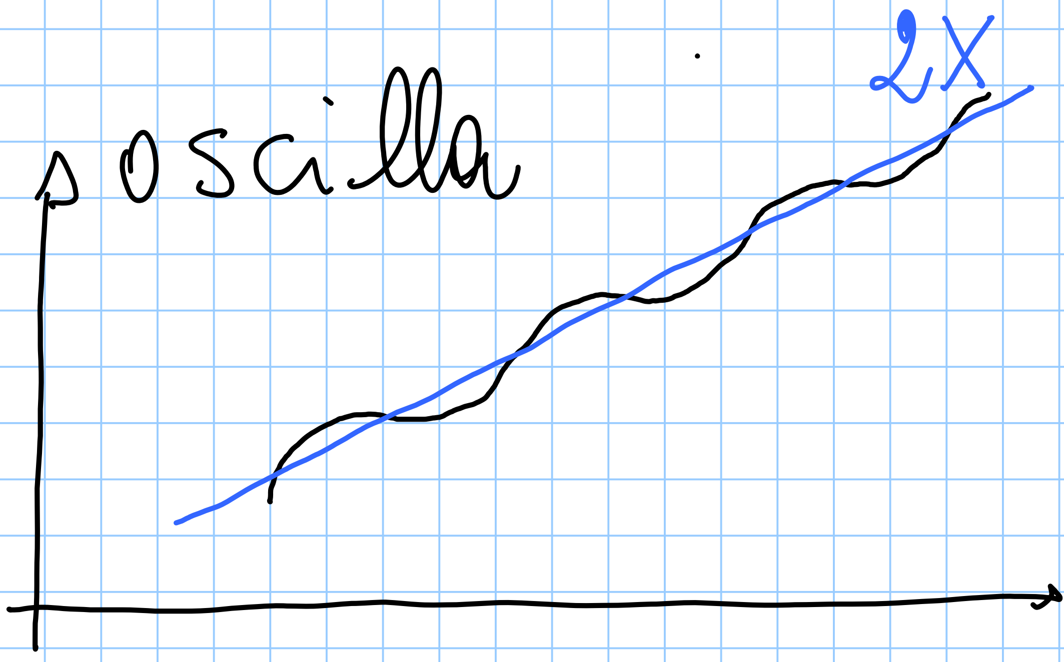
$$f(x) = 2x + \sin(x)$$

$$f'(x) = 2 + \cos(x) \geq 1 \quad \forall x$$

$$g(x) = 2 + \cos(x)$$

$$g'(x) = -\sin(x)$$

$f' = g$ oscilla



$$f(x) = -\frac{|x|}{2} + \operatorname{arcTg}(x)$$

- $\operatorname{dom}(f) = \mathbb{R}$
- $\lim_{x \to +\infty} f(x) = -\infty$

$$\lim_{x \to -\infty} f(x) = -\infty$$

$$\lim_{x \to +\infty} \frac{f(x)}{x} = \lim_{x \to +\infty} -\frac{|x|}{2x} + \frac{\operatorname{arcTg}(x)}{x}$$

$$= \lim_{x \to +\infty} -\frac{x}{2x} + \frac{\operatorname{arctg}(x)}{x} = -\frac{1}{2}$$

$$m_+ = -\frac{1}{2}$$

$$q_+ = \lim_{x \to +\infty} f(x) - m_+ x =$$

$$= \lim_{x \to +\infty} \cancel{-\frac{x}{2}} + \operatorname{arctg}(x) \cancel{- \left(-\frac{1}{2}\right) x}$$

$$= \lim_{x \to +\infty} \operatorname{arctg}(x) = \frac{\pi}{2}, \quad q_+ = \frac{\pi}{2}$$

ASINTOTO OBLIQUO $\underline{x \to +\infty}$

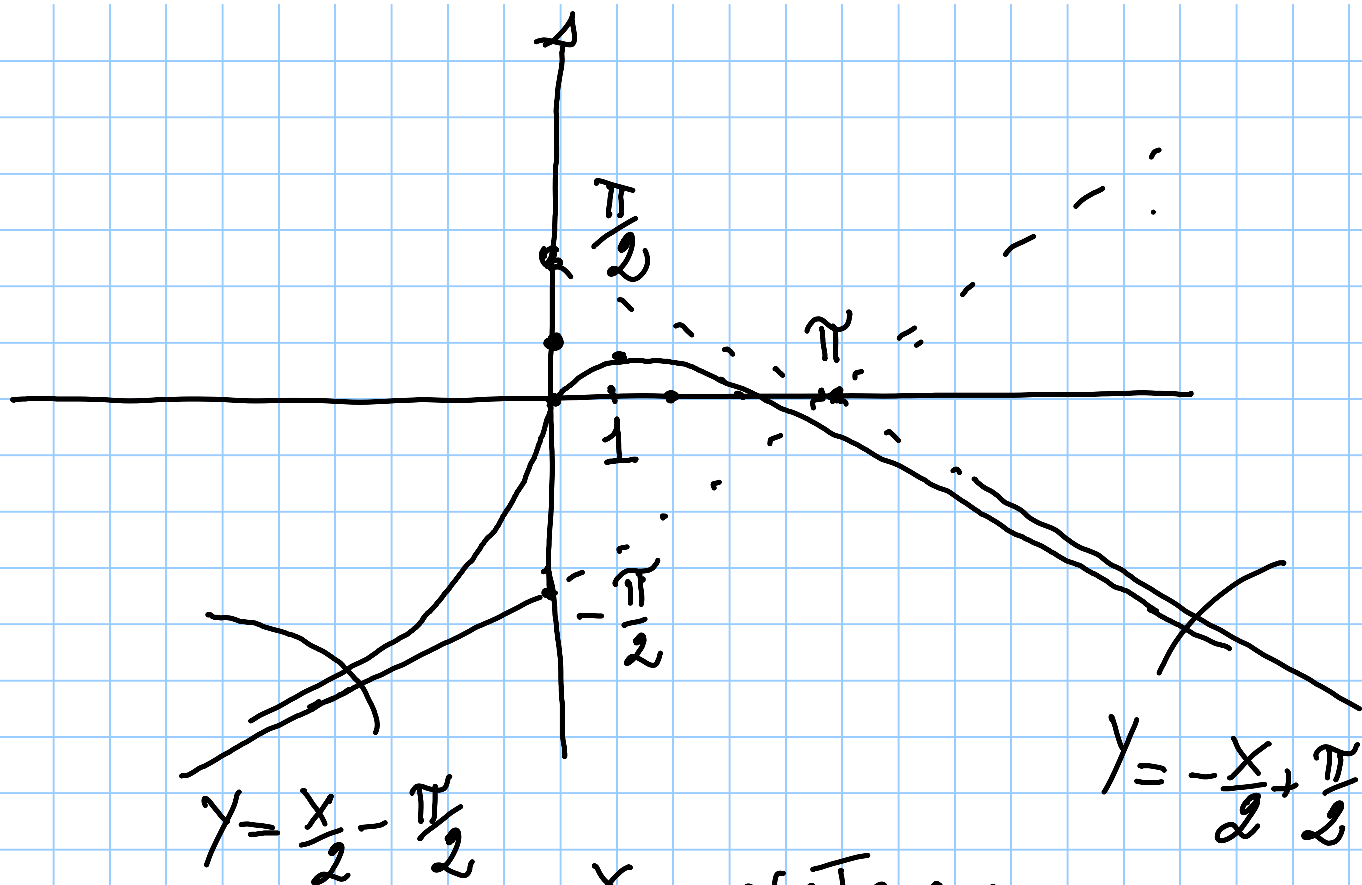
$$y = -\frac{1}{2} x + \frac{\pi}{2}$$

$\underline{x \to -\infty}$

$$y = \frac{x}{2} - \frac{\pi}{2}$$

$$f(x) = -\frac{|x|}{2} + \operatorname{arctg}(x) =$$

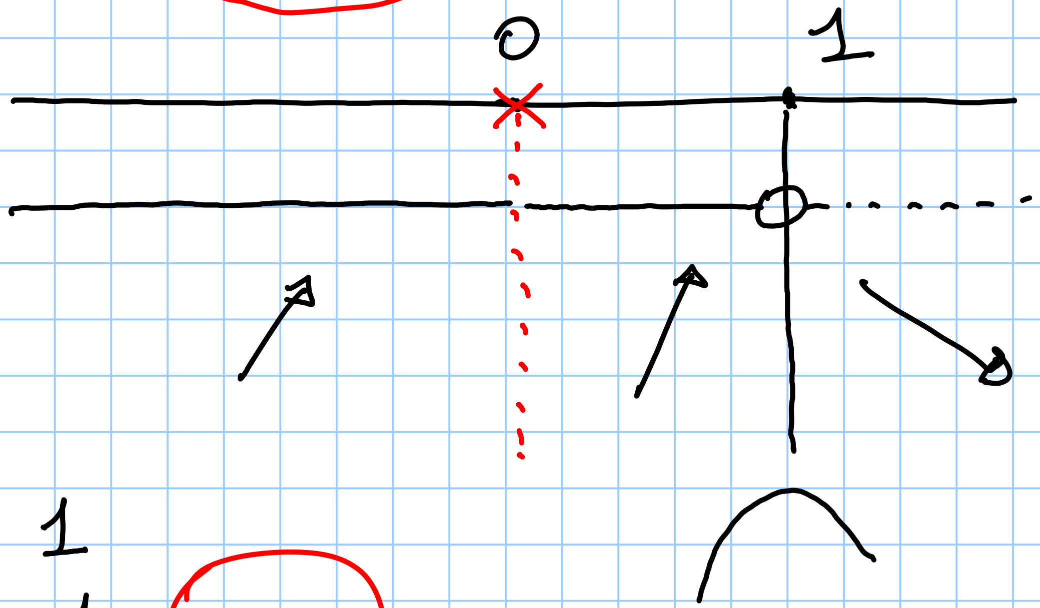
$$(x \to -\infty) \quad = \frac{x}{2} + \operatorname{arctg}(x) = \frac{x}{2} - \frac{\pi}{2} + o(1)$$



$$f(x) = \begin{cases} -\frac{x}{2} + \operatorname{arctg}(x), & x \geq 0 \\ \frac{x}{2} + \operatorname{arctg}(x), & x < 0 \end{cases}$$

$$f'(x) = \begin{cases} -\frac{1}{2} + \frac{1}{1+x^2}, & x > 0 \\ \frac{1}{2} + \frac{1}{1+x^2}, & x < 0 \end{cases}$$

$$= \begin{cases} \dfrac{(1-x^2)}{2(1+x^2)} \, , & x > 0 \\[2ex] \dfrac{3+x^2}{2(1+x^2)} > 0 \, , & x < 0 \end{cases}$$



$$1 - x^2 > 0$$

$$x^2 < 1$$

$$-1 < x < 1 \, , \quad x > 0$$

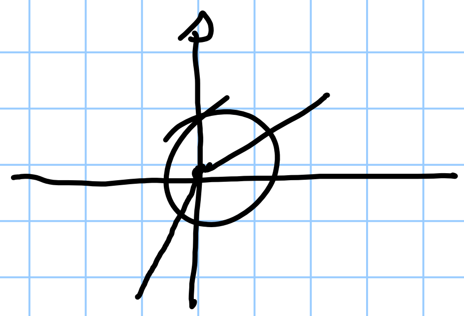
$x_0 = 1$
MASSIMO
LOCALE

$$f(1) = -\frac{1}{2} + \frac{\pi}{4} > 0$$

$$\lim_{x \to 0^+} f'(x) = \lim_{x \to 0^+} \frac{1-x^2}{2(1+x^2)} = \frac{1}{2}$$

$$\lim_{x \to 0^-} f'(x) = \lim_{x \to 0^-} \frac{3+x^2}{2(1+x^2)} = \frac{3}{2}$$

$x = 0$ PUNTO ANGOLOSO



? $$f(x) < -\frac{1}{2}x + \frac{\pi}{2}$$?

$$-\frac{|x|}{2} + \operatorname{arctg}(x) < -\frac{1}{2}x + \frac{\pi}{2}$$

$x > 0$

$$-\cancel{\frac{x}{2}} + \operatorname{arctg}(x) < \cancel{-\frac{1}{2}x} + \frac{\pi}{2}$$

VERO $\forall x > 0$

$$f(x) < -\frac{x}{2} + \frac{\pi}{2} \quad \forall x > 0$$

(Ex: $f(x) > \frac{x}{2} - \frac{\pi}{2}, \forall x < 0$)

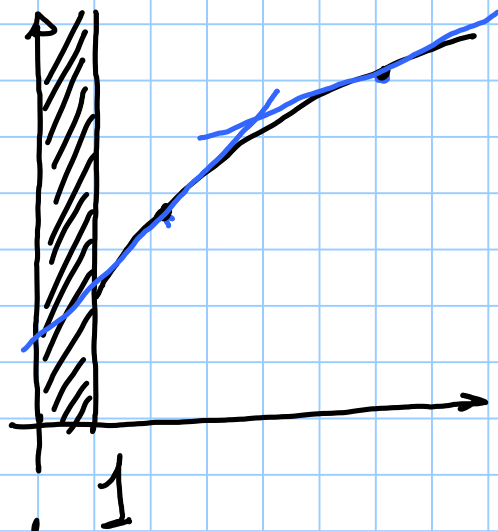
FUNZIONI CONVESSE/ CONCAVE

$f(x) = x^2$, $x \in [1, +\infty)$

$\lim_{x \to +\infty} f(x) = +\infty$, $f(1) = 1$

$f'(x) = 2x > 0 \ \forall x \geq 1$

$f(x) = \sqrt{x}$, $x \in [1, +\infty)$



$\lim_{x \to +\infty} f(x) = +\infty$, $f(1) = 1$

$f'(x) = \frac{1}{2\sqrt{x}} > 0 \ \forall x \geq 1$

$f' \nearrow \Longleftrightarrow$ CONVESSE

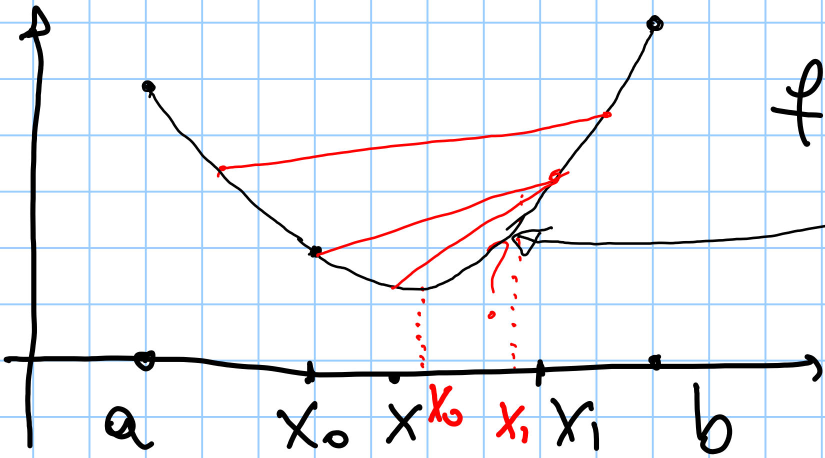
$f' \searrow \Longleftrightarrow$ CONCAVE

DEFINIZIONE $f:(a,b) \to \mathbb{R}$

f SI DICE CONVESSA (STRETTAMENTE)

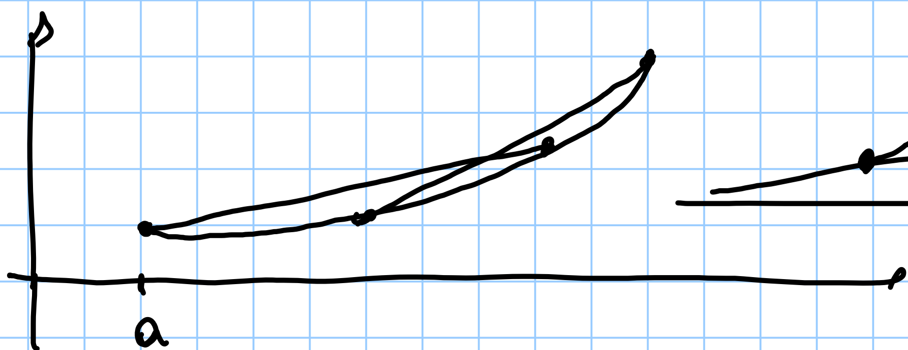
SE $f(x) \leq f(x_0) + \frac{f(x_1)-f(x_0)}{x_1 - x_0}(x-x_0)$ $(<)$

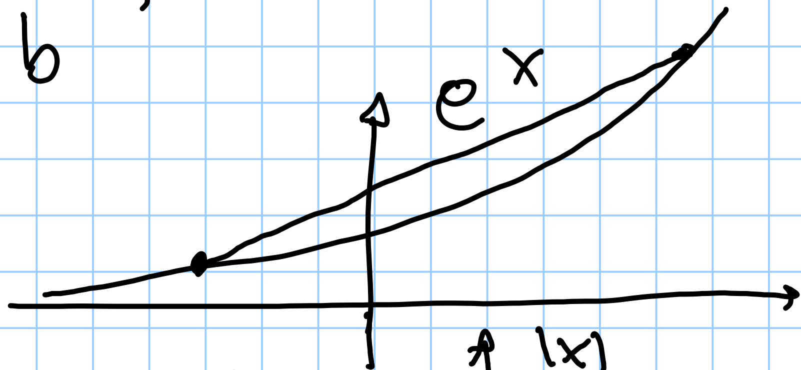
$\forall \quad a < x_0 < x < x_1 < b$

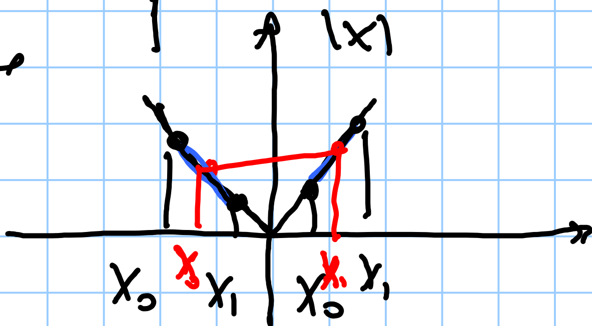


$f(x_0) + \frac{f(x_1)-f(x_0)}{x_1-x_0}(x-x_0)$

$x \in (x_0, x_1)$







TEOREMA

$f:(a,b)\longrightarrow \mathbb{R}$ è CONVESSA
(STRETTAMENTE)

$\Longleftrightarrow$ $(<)$

$$f(tx_1+(1-t)x_0)\leq tf(x_1)+(1-t)f(x_0)$$
$$\forall\ a<x_0<x_1<b$$
$$\forall\ t\in[0,1]$$

DIMOSTRAZIONE

COMBINAZIONE CONVESSA di x_0, x_1

$$x(t)=tx_1+(1-t)x_0\ ,\quad t\in[0,1]$$

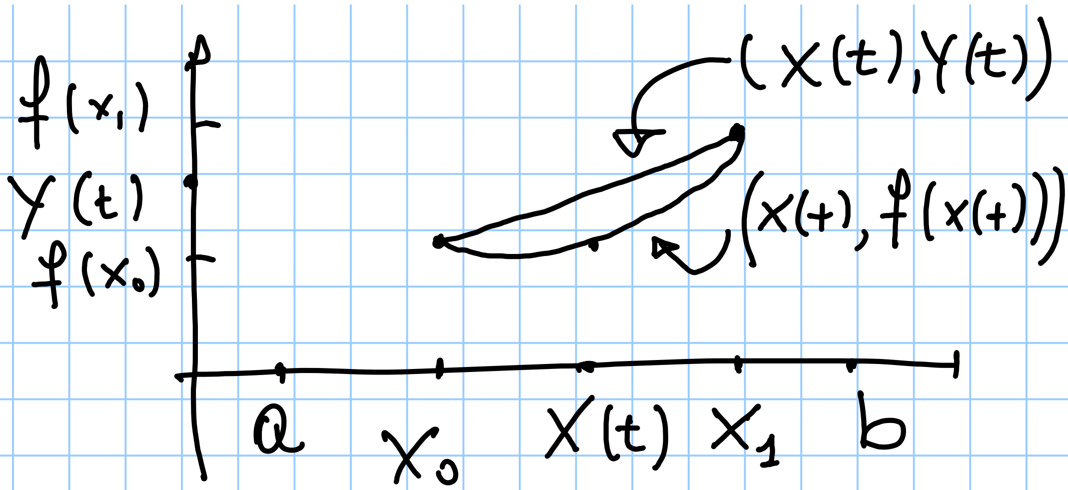
$$x(0)=x_0\ ,\quad x(1)=x_1$$

$x(t)$ si muove da x_0 a x_1, $t\in(0,1)$

$$y(t)=tf(x_1)+(1-t)f(x_0)$$

$y(t)$ si muove da $f(x_0)$ a $f(x_1)$

$(x(t),y(t))$ si muove da $(x_0,f(x_0))$ a $(x_1,f(x_1))$ lungo la retta per i due punti



$$x = t x_1 + (1-t) x_0 \Longrightarrow t = \frac{x - x_0}{x_1 - x_0}$$

$$f(x) \leq t f(x_1) + (1-t) f(x_0) =$$

$$= f(x_0) + \big(f(x_1) - f(x_0)\big) t$$

$$= f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0} (x - x_0)$$

$\blacksquare$

Questa definizione non RICHIEDE L'IPOTESI di DERIVABILITÀ.

DIMOSTRIAMO CHE

$f(x) = |x|$ è CONVESSA in $\mathbb{R}$

$$f(t x_1 + (1-t) x_0) \le t f(x_1) + (1-t) f(x_0)$$

$$\underbrace{|t x_1 + (1-t) x_0|}_{|x+y|} \le \underbrace{|t x_1|}_{|x|} + \underbrace{|(1-t) x_0|}_{|y|} =$$

$$= t|x_1| + (1-t)|x_0| =$$

$$= t f(x_1) + (1-t) f(x_0)$$

$\blacksquare$