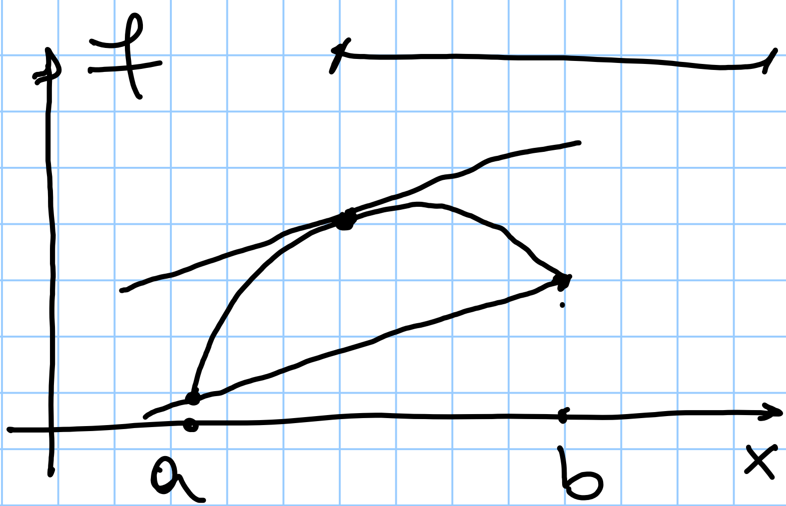
TEOREMA [LAGRANGE]

$f : [a,b] \longrightarrow \mathbb{R}$, CONTINUA in $[a,b]$

E DERIVABILE in (a,b). ALLORA

$\exists\; c \in (a,b)$:

$$Df(c) = \frac{f(b) - f(a)}{b-a}$$



DIMOSTRAZIONE

$$h(x) = f(x) - \left(f(a) + \frac{f(b)-f(a)}{b-a}(x-a)\right)$$

$h : [a,b] \to \mathbb{R}$, è CONTINUA in $[a,b]$, DERIVABILE (a,b) e

$$h(a) = 0 = h(b)$$

Dal TEOREMA di ROLLE

$$\exists\, c \in (a,b) : Dh(c) = 0$$

$$Df(c) - \frac{f(b)-f(a)}{b-a} = 0$$

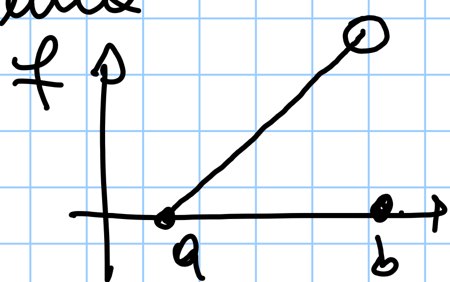
$\blacksquare$

f è CONTINUA in $[a,b)$

DERIVABILE in (a,b)

$\Longrightarrow$ il Teorema

NON VALE



TEOREMA [DERIVATE E MONOTONIA]

$f:(a,b)\to\mathbb{R}$, DERIVABILE in (a,b)

(i) $f'(x)\geq 0\ \forall x\in(a,b) \Longleftrightarrow$ f è CRESCENTE in (a,b)

(ii) $f'(x)>0\ \forall x\in(a,b) \Longrightarrow$ f è STRETTAMENTE CRESCENTE

(ii) $\not\Longleftarrow$ $f(x)=x^3,\ x\in\mathbb{R}$

<u>NOTA</u> $f\nearrow$ STRETTAMENTE ma $f'(0)=0$

DIMOSTRAZIONE

(i) $\Leftarrow$ f è CRESCENTE $\Leftrightarrow$

$$\forall x_0 \in (a,b), \ \frac{f(x)-f(x_0)}{x-x_0} \geq 0 \quad \forall x \neq x_0$$

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0} \geq 0, \forall x_0 \in (a,b)$$

$\Rightarrow$ Siano $a < x < y < b$

f è CONTINUA in $[x,y]$,
DERIVABILE in (x,y) $\Rightarrow$

dal T. di LAGRANGE

$$\frac{f(y)-f(x)}{y-x} = f'(c)$$

$c \in (x,y)$

$$\underbrace{f(y)-f(x)}_{\geq 0} = \underbrace{f'(c)}_{\geq 0} \cdot \underbrace{(y-x)}_{>0}$$

$$f(y) \geq f(x) \Longrightarrow f \nearrow$$

(ii) $\Longrightarrow$ STESSA dimostrazione del CASO (i)

$$a < x < y < b$$

$$f(y) - f(x) = \underbrace{f'(c)}_{>0}(y-x) > 0$$

$c \in (x,y)$

$$f(y) > f(x) \Longrightarrow$$

$f \nearrow$ STRETTAMENTE

$\blacksquare$

TEOREMA [CARATTERIZZAZIONE FUNZIONI DERIVABILI e COSTANTI]

$f : [a,b] \longrightarrow \mathbb{R}$, f CONTINUA in $[a,b]$ e DERIVABILE in (a,b),

SE $f'(x) = 0 \quad \forall x \in (a,b)$,

allora f è costante in $[a,b]$.

DIMOSTRAZIONE

APPLICHIAMO IL TEOREMA di LAGRANGE in $[a,x]$ con $x \in (a,b]$.

$$f(x) - f(a) = f'(c)(x-a) = 0$$

$$c \in (a,b)$$

$$f(x) = f(a) \quad \forall x \in [a,b] \qquad \blacksquare$$

APPLICAZIONE: DIMOSTRIAMO CHE

$$\operatorname{arctg}(x) = \frac{\pi}{2} - \operatorname{arctg}\left(\frac{1}{x}\right) \quad \forall x > 0$$

Per $x > 0$ poniamo

$$f(x) = \operatorname{arctg}(x) + \operatorname{arctg}\left(\frac{1}{x}\right)$$

$\lim_{x \to 0^+} f(x) = \frac{\pi}{2}$, quindi

estendiamo f per continuità

in $x = 0$, $f(0) = \frac{\pi}{2}$,

$f : [0, +\infty) \longrightarrow \mathbb{R}$, continua

in $[0, +\infty)$.

Per $x > 0$ si ha

$$f'(x) = \frac{1}{1+x^2} + \frac{1}{1+\frac{1}{x^2}} \cdot \left(-\frac{1}{x^2}\right) =$$

$$= \frac{1}{1+x^2} - \frac{1}{1+x^2} = 0$$

f è CONTINUA in $[0,b]$ $\forall b > 0$
DERIVABILE in $(0,b)$ e
$f'(x) = 0$ $\forall x \in (0,b)$

$\Rightarrow$ $f(x)$ è costante in $[0,b)$

$\Rightarrow$ $f(x) = \frac{\pi}{2}$ $\forall x \geq 0$

Ex Dimostrare che

$$\arccos(x) + \arcsin(x) = \frac{\pi}{2}, \quad \forall x \in [-1, 1]$$

TEOREMA [DERIVATE ED ESTREMI LOCALI]

$f : (a,b) \longrightarrow \mathbb{R}$, continua in (a,b), $x_0 \in (a,b)$, f derivabile in $(a,b) \setminus \{x_0\}$

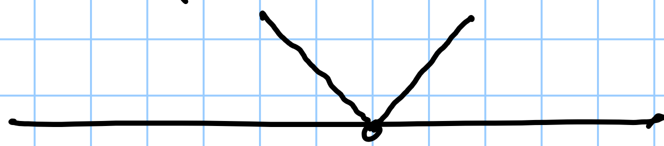
(i) Se $f'(x) > 0$ DEFINITIVAMENTE per $x \to x_0^+$

$f'(x) < 0$ DEFINITIVAMENTE per $x \to x_0^-$

Allora x_0 è un PUNTO di MINIMO LOCALE.

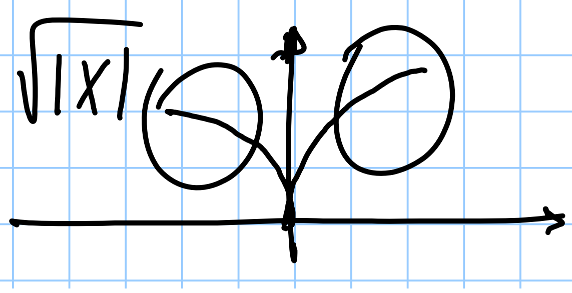
(ii) Se inoltre f è DERIVABILE in x_0, allora $f'(x_0) = 0$

$f(x) = |x|$



$x_0 = 0$ MINIMO LOCALE

$f(x) = \sqrt{|x|}$



DIMOSTRAZIONE

$f'(x)>0$ DEFINITIVAMENTE PER $x \to x_0^+$

OVVERO $\exists\, \delta^{(+)}>0$:

$f'(x)>0 \quad \forall x \in (x_0, x_0+\delta^{(+)})$

$f'(x)<0$ DEFINITIVAMENTE per $x \to x_0^-$

$\exists\, \delta^{(-)}>0$:

$f'(x)<0 \quad \forall x \in (x_0-\delta^{(-)}, x_0)$

$$\delta = \min\{\delta^{(-)}, \delta^{(+)}\}$$

$f \downarrow$ STRETTAMENTE in $(x_0-\delta, x_0)$

$f \uparrow$ STRETTAMENTE in $(x_0, x_0+\delta)$

$$\inf_{(x_0-\delta,x_0)} f = \lim_{x\to x_0^-} f(x) = f(x_0)$$

monotonia

continuità

$$\inf_{(x_0,x_0+\delta)} f = \lim_{x\to x_0^+} f(x) = f(x_0)$$

$$f(x) \geq f(x_0) \quad \forall x \in (x_0-\delta, x_0)$$

$$\forall x \in (x_0, x_0+\delta)$$

$$f(x) \geq f(x_0) \quad \forall x \in (x_0-\delta, x_0+\delta)$$

$\Longrightarrow x_0$ MINIMO LOCALE

(ii) SE f è DERIVABILE

in x_0 $\Longrightarrow$ T. di FERMAT

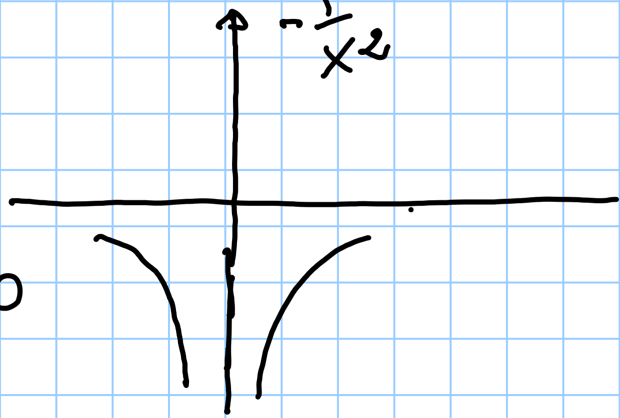
$$f'(x_0) = 0$$

$\blacksquare$

CONTROESEMPI

SE f non è continua in x_0 il TEOREMA NON VALE

$$f(x) = -\frac{1}{x^2}$$

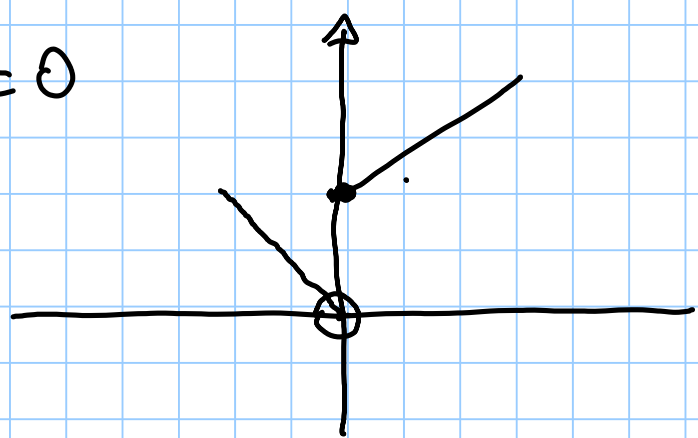


$$\nexists \min_{\mathbb{R}\setminus\{0\}} f \;;\; \inf_{\mathbb{R}\setminus\{0\}} f = -\infty$$

$$f(x) = \begin{cases} -x, & x < 0 \\ 1+x, & x \geq 0 \end{cases} \;;\; \lim_{x \to 0^+} f(x) = 1 \quad \lim_{x \to 0^-} f(x) = 0$$

$$f(0) = 1$$

$$\nexists \min_{\mathbb{R}} f \;,\; \inf_{\mathbb{R}} f = 0$$



TEOREMA [CONTINUITÀ DELLA DERIVATA PRIMA]

$f : [a,b] \to \mathbb{R}$, $x_0 \in [a,b]$,

f DERIVABILE in $[a,b] \setminus \{x_0\}$,

CONTINUA in $[a,b]$

SE $\exists$ (FINITO O INFINITO)

$$\lim_{x \to x_0} f'(x) \quad \left(\begin{array}{l} x_0 = a, \ x \to a^+ \\ x_0 = b, \ x \to b^- \end{array} \right)$$

Allora $$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0} f'(x)$$

SE il $\lim_{x \to x_0} f'(x) \ \nexists$

NON SI PUO' DIRE NULLA IN GENERALE

su $$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

ESEMPIO

$$f(x) = \begin{cases} x^2 \cos\left(\frac{1}{x}\right) & , \ x \neq 0 \\ 0 & , \ x = 0 \end{cases}$$

$$\lim_{x\to 0} f(x) = 0 = f(0),$$

f è continua in $\mathbb{R}$

DERIVABILE in $\mathbb{R}$

$$f'(x) = 2x\cos\left(\frac{1}{x}\right) + \sin\left(\frac{1}{x}\right), \quad x \neq 0$$

$$f'(0) = \lim_{x\to 0} \frac{f(x)-f(0)}{x-0} =$$

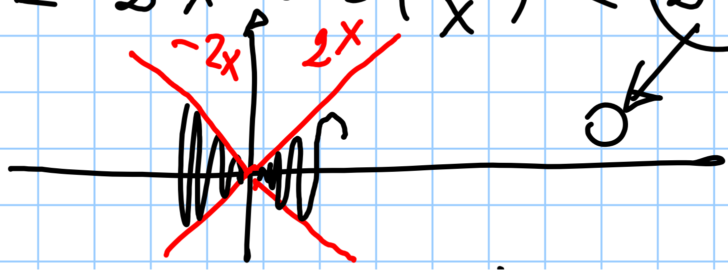
$$= \lim_{x\to 0} x\cos\left(\frac{1}{x}\right) = 0$$

$$\lim_{x\to 0} f'(x) \ ? \ \nexists$$

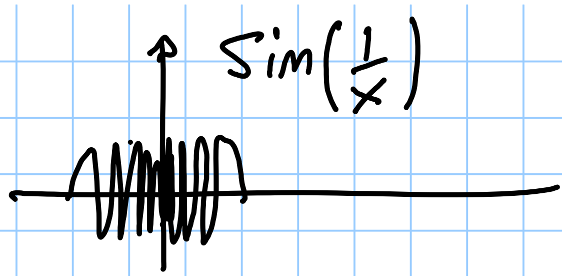
$$\underbrace{2x\cos\left(\frac{1}{x}\right)}_{\downarrow\, 0} + \sin\left(\frac{1}{x}\right)$$

$$-2x \le 2x\cos\left(\frac{1}{x}\right) \le 2x$$

$-2x \to 0$, $x \to 0$; $2x \to 0$



$$\lim_{x \to 0} \sin\left(\frac{1}{x}\right) \nexists$$



$$\sin(a_n) \longrightarrow 0 \qquad a_n \longrightarrow +\infty$$

$$\sin(b_n) \longrightarrow 1 \qquad b_n \longrightarrow +\infty$$

DIMOSTRAZIONE

Sia $\{x_n\}$ una SUCCESSIONE,

$x_n \to x_0^+$, $\quad \{x_n\} \subset (a,b)$

Dal T. di LAGRANGE, $\forall n \in \mathbb{N}$

$$f(x_n) - f(x_0) = f'(c_n)(x_n - x_0)$$

$$c_n \in (x_0, x_n)$$

$$x_0 < c_n < x_n \Longrightarrow c_n \xrightarrow[n \to +\infty]{} x_0$$

$$\Longrightarrow \lim_{n \to +\infty} \frac{f(x_n) - f(x_0)}{x_n - x_0} = \lim_{n} f'(c_n)$$

PER IPOTESI $\lim_{x \to x_0^+} f'(x) = L$

$$\Rightarrow \lim_{n} f'(c_n) = L \Rightarrow$$

$$\lim_{n \to +\infty} \frac{f(x_n) - f(x_0)}{x_n - x_0} = L$$

$$\forall \{x_n\} \subset (a,b) : \ x_n \xrightarrow[n \to +\infty]{} x_0^+$$

OVVERO $\lim_{x \to x_0^+} \frac{f(x) - f(x_0)}{x - x_0} = L$

$\blacksquare$

GRAFICI DI FUNZIONI

$f(x)$, dominio di definizione

limiti agli estremi del dominio,

ASINTOTI, DERIVABILITÀ,

eventuali punti di NON DERIVABILITÀ,

intervalli di MONOTONIA,

MASSIMI e MINIMI LOCALI,

GRAFICO QUALITATIVO

max f / sup f

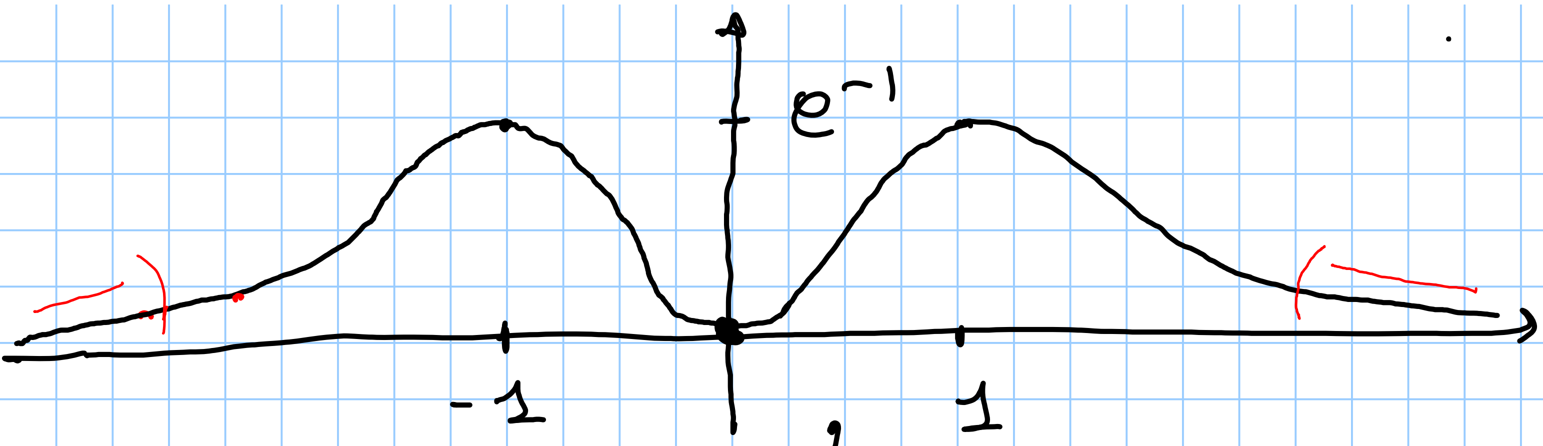
min f / Inf f

$$f(x) = x^2 e^{-x^2}, \quad dom(f) = \mathbb{R}$$

f è PARI $\quad f(-x) = f(x)$

$$f(0) = 0$$

$$\lim_{x \to +\infty} f(x) = 0^+, \quad \lim_{x \to -\infty} f(x) = 0^+$$



$f(x) = x^2 e^{-x^2}$, DERIVABILE in dom(f)

$$f'(x) = 2x\, e^{-x^2} + x^2 e^{-x^2}(-2x)$$

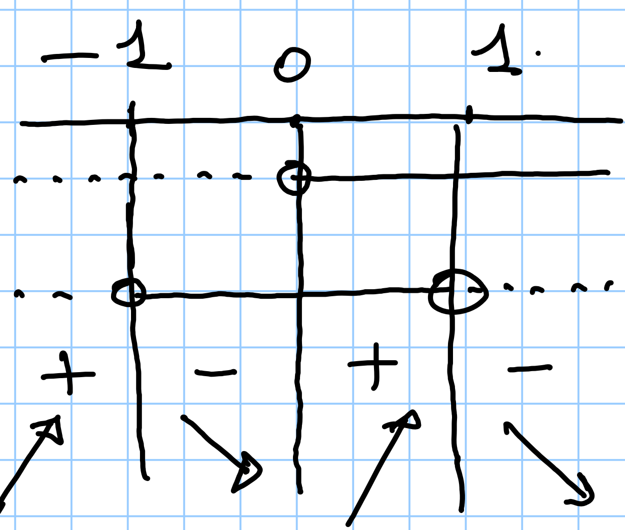
$$= \underbrace{2x\, e^{-x^2}(1-x^2)}_{\text{SEGNO ?}}$$

$x(1-x^2) \geq 0$

x

$1-x^2 \geq 0$, $x^2 \leq 1$

$-1 \leq x \leq 1$



$$f(\pm 1) = e^{-1}$$

$$f(0) = 0$$

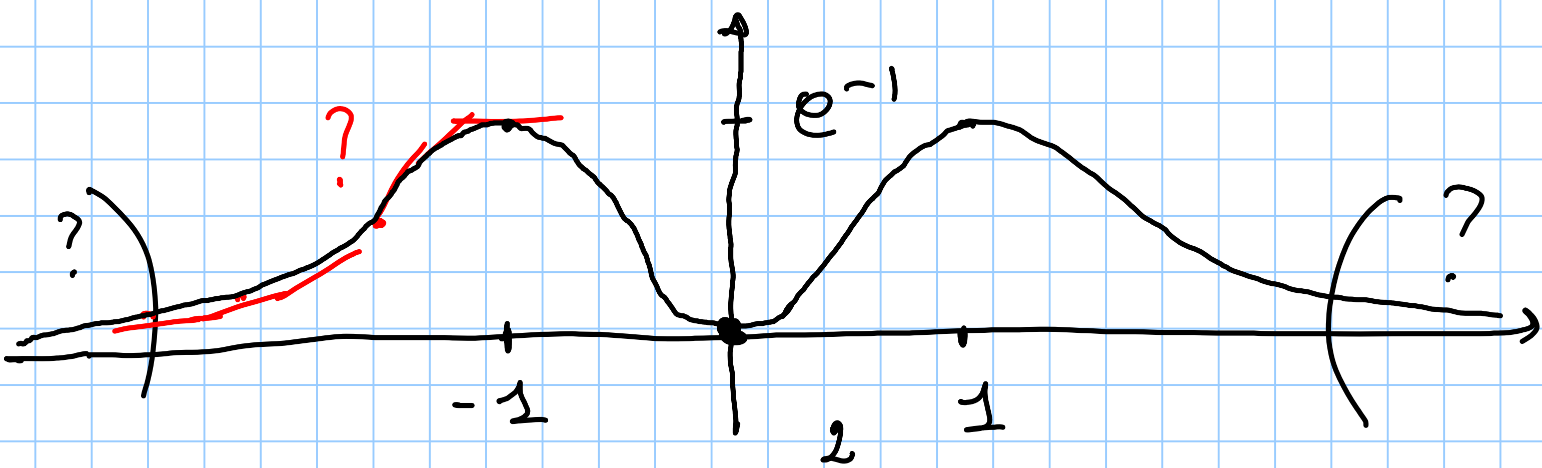
$x_0 = -1$ MASSIMO locale

$x_0 = 0$ MINIMO locale

$x_0 = 1$ MASSIMO locale

$$\min_{\mathbb{R}} f = 0 = f(0)$$

$$\max_{\mathbb{R}} f = e^{-1} = f(1)$$



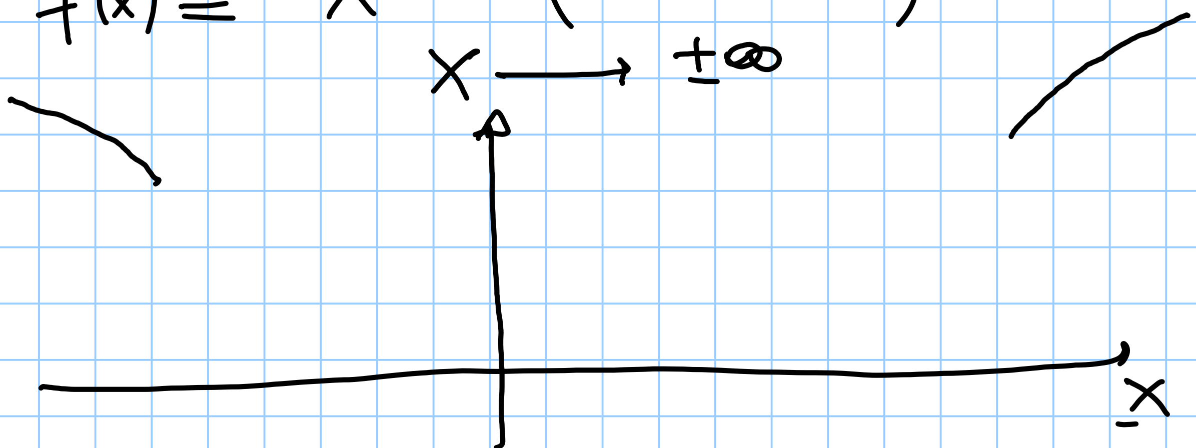
$$f(x) = \left(x^2+1-|3x-1|\right)^{\frac{1}{3}}$$

$$\mathrm{dom}(f) = \mathbb{R}$$

$$\lim_{x \to +\infty} f(x) = +\infty$$

$$\lim_{x \to -\infty} f(x) = +\infty$$

$$f(x) \underset{x \to \pm\infty}{=} x^{2/3}\left(1+o(1)\right)$$



$$f(x) = \begin{cases} \left(x^2+1-(3x-1)\right)^{\frac{1}{3}}, & x \geq \frac{1}{3} \\ \left(x^2+1+(3x-1)\right)^{\frac{1}{3}}, & x < \frac{1}{3} \end{cases}$$

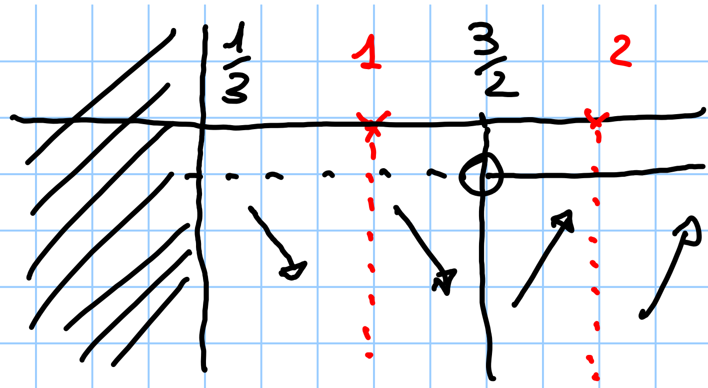
$$= \begin{cases} \left(x^2-3x+2\right)^{\frac{1}{3}}, & x \geq \frac{1}{3} \\ \left(x^2+3x\right)^{\frac{1}{3}}, & x < \frac{1}{3} \end{cases}$$

$$f'(x) = \begin{cases} \frac{1}{3}\,\frac{2x-3}{(x^2-3x+2)^{\frac{2}{3}}}, & x > \frac{1}{3} \\ & x \neq 1,2 \\ \frac{1}{3}\,\frac{2x+3}{(x^2+3x)^{\frac{2}{3}}}, & x < \frac{1}{3} \\ & x \neq 0,-3 \end{cases}$$

$$\mathrm{dom}(f') = \mathrm{dom}(f) \setminus \left\{\frac{1}{3}, -3, 0, 1, 2\right\}$$

$$\frac{2x-3}{\underbrace{(x^2-3x+2)^{\frac{2}{3}}}_{\geq 0}} \geq 0 \iff x \geq \frac{3}{2}, \quad \boxed{x > \frac{1}{3}}$$

$$x \neq 1,2$$



$x_0 = 3/2$ MINIMO LOCALE

$$f\left(\frac{3}{2}\right) = \left(-\frac{1}{4}\right)^{\frac{1}{3}}$$