DIM.

$$D(f+g)(x_0) = Df(x_0) + Dg(x_0)$$

FACILE

$$D(fg)(x_0) =$$

$$= Df(x_0) \cdot g(x_0) + f(x_0) \cdot Dg(x_0)$$

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + o(x-x_0)$$

$$g(x) = g(x_0) + g'(x_0)(x-x_0) + o(x-x_0)$$

$$x \to x_0$$

$$f(x)g(x) =$$

$$\left(f(x_0) + f'(x_0)(x-x_0) + o(x-x_0)\right) \cdot$$

$$\left(g(x_0) + g'(x_0)(x-x_0) + o(x-x_0)\right) =$$

$$= f(x_0)g(x_0) + f(x_0)g'(x_0)(x-x_0) + o(x-x_0)$$

$$+ f'(x_0)g(x_0)(x-x_0) + o(x-x_0) +$$

$$f'(x_0)g'(x_0)(x-x_0)^2 + o(x-x_0)\cdot o(x-x_0)$$

$$\frac{(x-x_0)^{\cancel{2}}}{\cancel{(x-x_0)}} \xrightarrow[x \to x_0]{} 0$$

$= o(x-x_0)$ $\qquad$ $= o(x-x_0)$

$$= f(x_0)g(x_0) + \boxed{\big(f'(x_0)g(x_0) + f(x_0)g'(x_0)\big)}(x-x_0) + o(x-x_0)$$

DIFFERENZIABILITA' per

$$h(x) = f(x)g(x)$$

$$D(fg)(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0)$$

$h(x) = f(x)g(x)$, $\quad h(x) = h(x_0) + \lambda(x-x_0) + o(x-x_0)$

$$D(f \circ g)(x_0) = Df(g(x_0)) \cdot Dg(x_0)$$

$$f(y) - f(y_0) = Df(y_0)(y-y_0) + o(y-y_0)$$

$$y \to y_0$$

$$y_0 = g(x_0)$$

$$y = g(x)$$

$$f(g(x)) - f(g(x_0)) =$$

$$Df(g(x_0))(g(x) - g(x_0)) + o(g(x) - g(x_0))$$

$$\left\{ g(x) - g(x_0) = Dg(x_0)(x-x_0) + o(x-x_0) \right\}$$

$$= Df(g(x_0))(Dg(x_0)(x-x_0) + o(x-x_0))$$

$$+ \boxed{o(Dg(x_0)(x-x_0) + o(x-x_0))} = o(x-x_0)$$

$$= Df(g(x_0))Dg(x_0)(x-x_0) + o(x-x_0)$$

DIFFERENZIABILITA' per

$f \circ g(x)$ in x_0

$$D(f \circ g)(x_0) = Df(g(x_0))\,Dg(x_0)$$

$$\left(\frac{f}{g}\right)' = \frac{f'g - f g'}{g^2}$$

$$\left(Tg(x)\right)' = \left(\frac{\sin(x)}{\cos(x)}\right)' =$$

$$= \frac{\cos^2(x) - (-\sin^2(x))}{\cos^2(x)} =$$

$$= \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)}$$

$$= 1 + Tg^2(x)$$

$$D\left(\frac{1}{\sqrt[4]{1+x^2}}\right) = D\left((1+x^2)^{-\frac{1}{4}}\right) =$$

$$-\frac{1}{4}\,(1+x^2)^{-\frac{1}{4}-1}\cdot(2x) = -\frac{1}{2}\,\frac{x}{(1+x^2)^{5/4}}$$

TEOREMA [DERIVABILITA' FUNZIONE INVERSA]

$f : (a,b) \to \mathbb{R}$, INIETTIVA

E DERIVABILE in $x_0 \in (a,b)$ e sia

$g = f^{-1}$ la funzione inversa di f

SE $Df(x_0) \neq 0$, ALLORA

$g(y)$ E' DERIVABILE in $y_0 = f(x_0)$

e si ha: $$Dg(y_0) = \frac{1}{Df(g(y_0))}.$$

DIMOSTRAZIONE

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = Df(x_0) \neq 0$$

$$\lim_{x \to x_0} \frac{x - x_0}{f(x) - f(x_0)} = \frac{1}{Df(x_0)}$$

$$\lim_{y \to y_0} \frac{g(y) - g(y_0)}{y - y_0} =$$

$x = g(y)$ $\quad$ $y = f(x)$

$x_0 = g(y_0)$ $\quad$ $y_0 = f(x_0)$

$$\lim_{x \to x_0} \frac{x - x_0}{f(x) - f(x_0)} = \frac{1}{Df(x_0)} =$$

$x_0 = g(y_0)$

$$= \frac{1}{Df(g(y_0))}$$

$x = g(y) \to g(y_0) = x_0$

CONTINUITA' funzione INVERSA

$$f(g(y)) = y \qquad y \in Im(f)$$

SE SAPESSIMO CHE

g è DERIVABILE $\Rightarrow$

$$Df(g(y_0)) \cdot Dg(y_0) = 1$$

$$Dg(y_0) = \frac{1}{Df(g(y_0))}$$

$$g(y) = \sqrt{y} \quad , y \in [0,+\infty)$$

$$g = f^{-1} \quad , \quad f(x) = x^2, \quad x \in (0,+\infty)$$

$$g'(y) = \frac{1}{f'(x)}\Big|_{x=g(y)} =$$

NOTA

$\mathrm{dom}\, g' \subset \mathrm{dom}\, g$

$(0,+\infty) \subset [0,+\infty)$

$$= \frac{1}{2x}\Big|_{x=g(y)} = \frac{1}{2x}\Big|_{x=\sqrt{y}} = \frac{1}{2\sqrt{y}}, \quad y \in (0,+\infty)$$

$$g(y) = \operatorname{arctg}(y), \quad y \in \mathbb{R}$$

$$g = f^{-1}, \quad f(x) = \operatorname{Tg}(x) \quad x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$$g'(y) = \frac{1}{f'(x)\big|_{x=g(y)}} = \frac{1}{1+\operatorname{Tg}^2(x)}\Big|_{x=\operatorname{arcTg}(y)}$$

$$= \frac{1}{1+\operatorname{Tg}^2(x)}\Big|_{\operatorname{Tg}(x)=y} = \frac{1}{1+y^2}, \quad y \in \mathbb{R}$$

$$g(y) = \arccos(y), \quad y \in [-1,1]$$

$$f(x) = \cos(x) \quad x \in [0,\pi]$$

$$g'(y) = \frac{1}{-\sin(x)}\Bigg|_{x=\arccos y} =$$

$$= \frac{1}{-\sin(x)}\Bigg|_{\cos(x)=y}$$

$$\sin(x) = \pm\sqrt{1-\cos^2(x)}$$

$$x \in [0,\pi] \longrightarrow \sin(x) > 0$$

$$\sin(x) = \sqrt{1-\cos^2(x)}$$

$$g'(y) = \frac{1}{-\sqrt{1-\cos^2(x)}}\Bigg|_{\cos(x)=y} =$$

$$= \frac{-1}{\sqrt{1-y^2}} \; ; \quad y \in (-1,1)$$

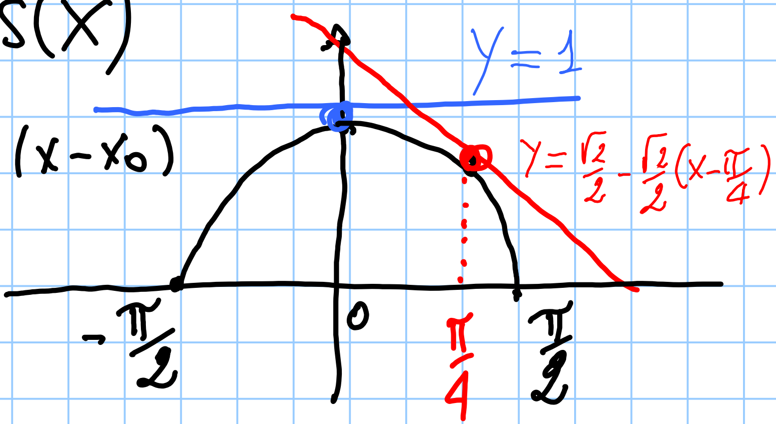
! $\underline{\mathrm{dom}(g')} \subset \underline{\mathrm{dom}(g)}$!

PUNTI di' NON DERIVABILITÀ

$$\begin{cases} g(y) = \sqrt{y}, & y \in [0,+\infty) \\ g'(y) = \dfrac{1}{2\sqrt{y}}, & y \in (0,+\infty) \end{cases}$$

$$\mathrm{dom}(g') \subset \mathrm{dom}(g)$$

$$f(x) = \cos(x)$$



$$f(x) = f(x_0) + f'(x_0)(x-x_0) + o(x-x_0)$$

$$x_0 = \frac{\pi}{4}$$

$$\cos(x) = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right) + o\left(x - \frac{\pi}{4}\right) \quad x \to \frac{\pi}{4}$$

$$y = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right),$$ retta tangente

$$x_0 = 0$$

$$\cos(x) = 1 + o(x) \quad x \to 0$$

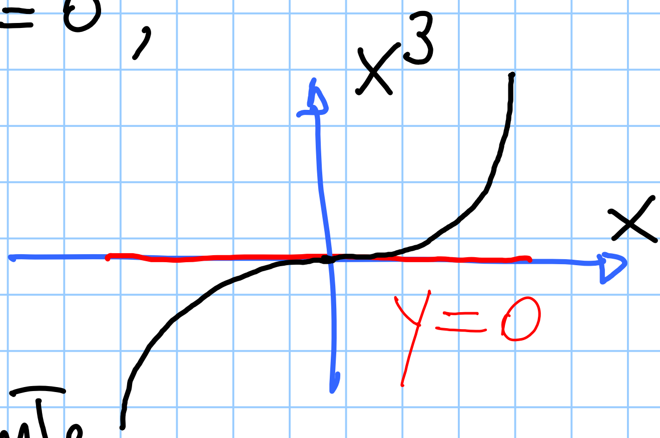
$$y = 1$$ retta tangente

$f(x) = x^3$, $f'(0) = 0$,

$f(x) = 0 + 0 \cdot x + o(x)$

$y = 0$

retta tangente



DEFINIZIONE

$f : [a,b] \longrightarrow \mathbb{R}$ Si dice che f è DERIVABILE da DESTRA in a, se $\exists$ finito

$$(L_1) \quad \lim_{x \to a^+} \frac{f(x) - f(a)}{x - a}$$

DERIVABILE da SINISTRA in b, se $\exists$ finito

$$(L_2) \quad \lim_{x \to b^-} \frac{f(x) - f(b)}{x - b}$$

$$Df^{(+)}(a), \quad f'_{(+)}(a)$$

$$Df^{(-)}(a), \quad f'_{(-)}(a)$$

NOTA: $f(x)$ è derivabile

in $x_0 \in (a,b) \iff$

$\exists\, D^{(+)}f(x_0)$, $\exists\, D^{(-)}f(x_0)$,

$$D^{(+)}f(x_0) = D^{(-)}f(x_0)$$

TEOREMA [CONTINUITA' della DERIVATA PRIMA]

Se f è continua in (a,b),

$x_0 \in (a,b)$, DERIVABILE in $(a,b) \setminus \{x_0\}$

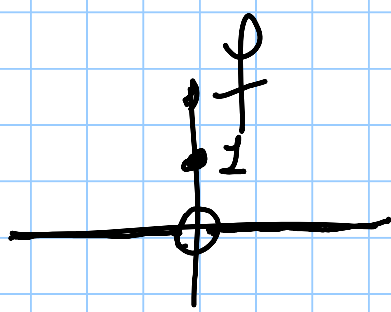
e $\lim_{x \to x_0^{\pm}} f'(x) = L_{\pm}$

ALLORA

$$\lim_{x \to x_0^{\pm}} \frac{f(x) - f(x_0)}{x - x_0} = L_{\pm}$$

ESEMPIO

$$f(x)=\begin{cases} 0 & , x\neq 0 \\ 1 & , x=0 \end{cases}$$



$$\lim_{x\to 0} f(x) = 0 \neq f(0) = 1$$

E' un esempio di discontinuità ELIMINABILE: IL LIMITE di $f(x)$, $x \to x_0$ esiste finito ma è DIVERSO da $f(x_0)$

DAL Teorema di continuità della derivata prima,

Se f è DERIVABILE e se

$\lim_{x \to x_0} f'(x) = L$ ALLORA

$$L = f'(x_0)$$

OVVERO SE $g = f'(x)$, CIOE'

SE g è una funzione DERIVATA,

NON HA DISCONTINUITÀ ELIMINABILI.

IN PARTICOLARE

SE f è DERIVABILE

in $(a,b) \setminus \{x_0\}$ e

continua in (a,b)

e se $\exists \lim_{x \to x_0^\pm} f'(x_0)$

ALLORA per calcolare

$$\lim_{x \to x_0^\pm} \frac{f(x) - f(x_0)}{x - x_0}$$

basta calcolare $\lim_{x \to x_0^\pm} f'(x_0)$

DEFINIZIONE

$f:(a,b)\to\mathbb{R}$, $x_0\in(a,b)$

Se f è continua in x_0 e derivabile da sinistra e da destra in x_0 e se $D^{(+)}f(x_0)\neq D^{(-)}f(x_0)$

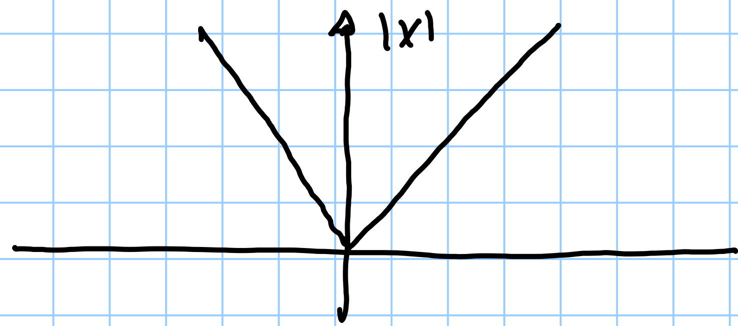
allora $(x_0, f(x_0))$ si dice PUNTO ANGOLOSO.

In sostanza: la retta tangente da sinistra è diversa dalla retta tangente da destra.

$$f(x) = |x| \quad , \quad f(x) = \begin{cases} x & , x \geq 0 \\ -x, & x < 0 \end{cases}$$

$$f'(x) = \begin{cases} 1 , & x > 0 \\ -1, & x < 0 \end{cases}$$

$$D^{(+)} f(0) = 1 \neq -1 = D^{(-)} f(0)$$



$$f(x) = |\log(x-1)| =$$

$$= \begin{cases} \log(x-1) , & x-1 \geq 1 \\ -\log(x-1) , & 0 < x-1 < 1 \end{cases}$$

$$= \begin{cases} \log(x-1) , & x \geq 2 \\ -\log(x-1) , & 1 < x < 2 \end{cases}$$

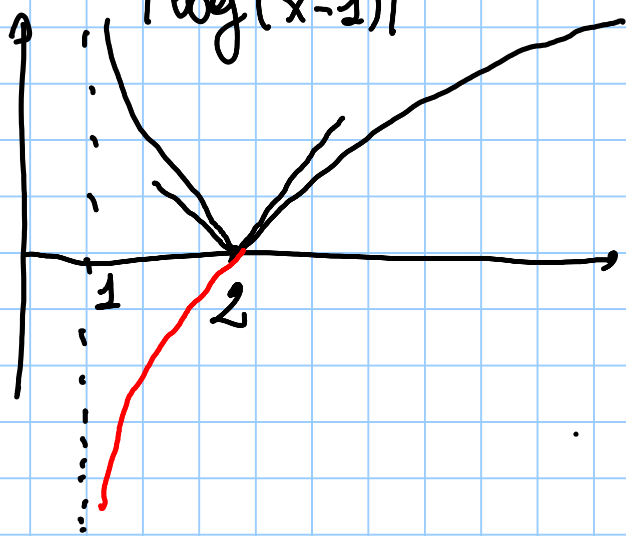
$$Df(x) = \begin{cases} \frac{1}{x-1} , & x > 2 \\ -\frac{1}{x-1} , & x \in (1,2) \end{cases}$$

$$D^{(+)}f(2) = \lim_{x \to 2^+} Df(x) = 1$$

$$D^{(-)}f(2) = \lim_{x \to 2^-} Df(x) = -1$$

PUNTO ANGOLOSO

$|\log(x-1)|$



DEFINIZIONE

Sia $f : [a,b] \to \mathbb{R}$, $x_0 \in [a,b]$.

Se f è continua in x_0 e se

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = +\infty$$

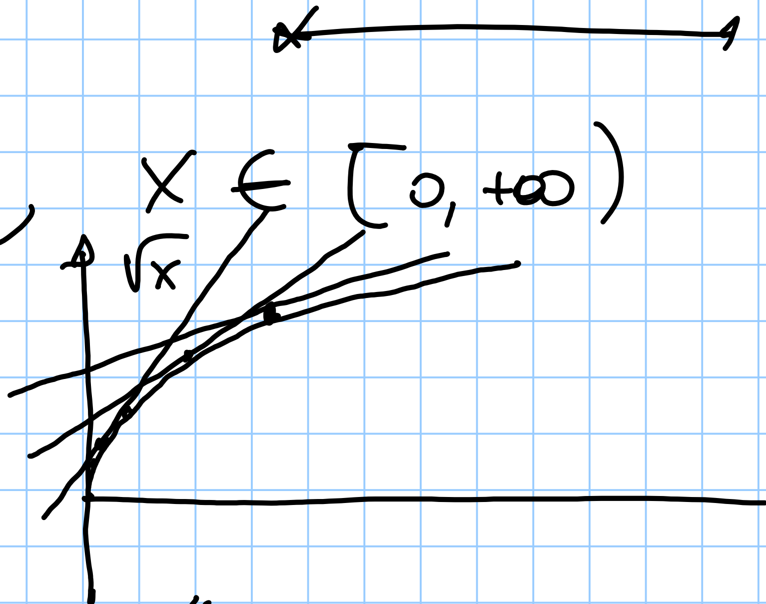
o se

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = -\infty$$

x_0 si dice PUNTO di TANGENZA

VERTICALE.

$$f(x) = \sqrt{x}, \quad x \in [0, +\infty)$$



$$\lim_{x \to 0^+} \frac{\sqrt{x} - 0}{x - 0} = \lim_{x \to 0^+} \frac{1}{\sqrt{x}} = +\infty$$

"LA RETTA TANGENTE" è

$$r : \{(x, y) : x = 0\}$$

$$f'(x) = \frac{1}{2\sqrt{x}}, \quad x \in (0, +\infty)$$

$$\lim_{x \to 0^+} f'(x) = +\infty$$

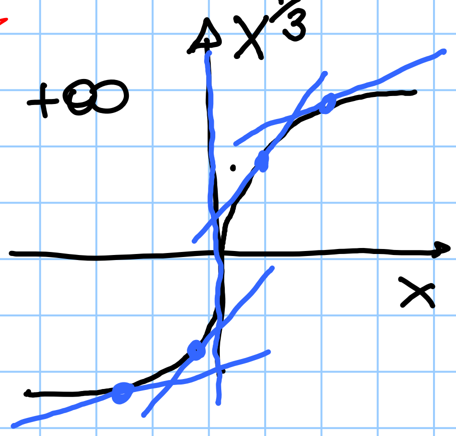
CONTINUITÀ DERIVATA PRIMA

$$f(x) = x^{\frac{1}{3}} \qquad x \in \mathbb{R}$$

$$f'(x) = \frac{1}{3} x^{-2/3} \qquad x \in \mathbb{R} \setminus \{0\}$$

$$\lim_{x \to 0} f'(x) = +\infty$$

$$\lim_{x\to 0} \frac{x^{\frac{1}{3}}}{x} = \lim_{x\to 0} \frac{1}{x^{2/3}} = +\infty$$



NOTA

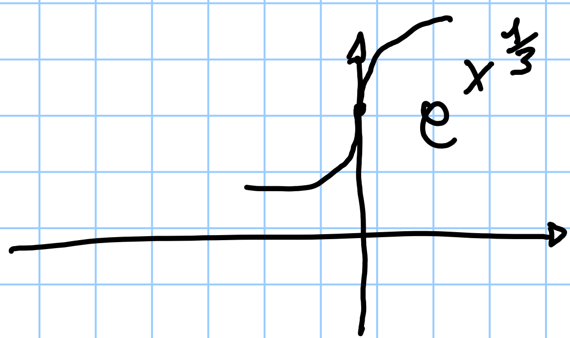
$$f(x) = \sin\left(x^{\frac{1}{3}}\right)$$

$$\sin\left(x^{\frac{1}{3}}\right) = x^{\frac{1}{3}} + o\left(x^{\frac{1}{3}}\right), \quad x \to 0$$

$$\sin(y) = y + o(y), \quad y \to 0$$

$$f(x) = e^{x^{\frac{1}{3}}} \underset{x\to 0}{=} 1 + x^{1/3} + o\left(x^{\frac{1}{3}}\right)$$

$$e^y = 1 + y + o(y), \quad y \to 0$$

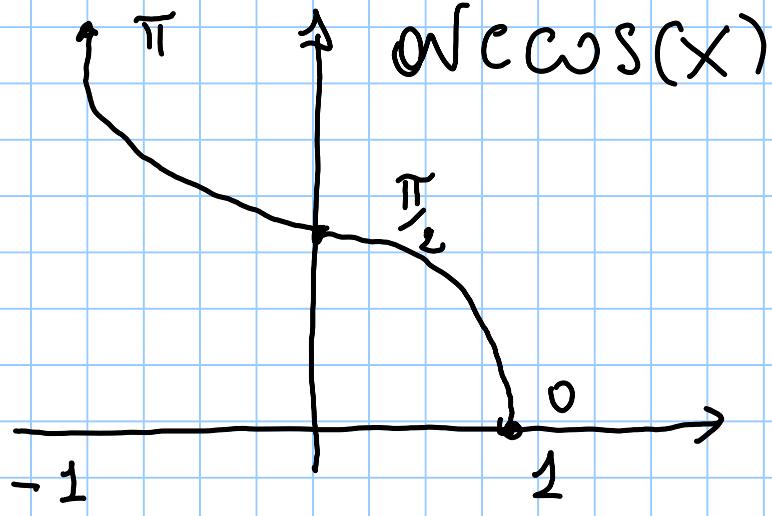


$$f(x) = \arccos(x) \quad , x \in [-1,1]$$

$$f'(x) = \frac{-1}{\sqrt{1-x^2}} \quad , x \in (-1,1)$$

$$\lim_{x \to (-1)^+} f'(x) = -\infty$$

$$\lim_{x \to 1^-} f'(x) = -\infty$$



DEFINIZIONE

Sia $f:(a,b) \to \mathbb{R}$, $x_0 \in (a,b)$

Se f è CONTINUA in x_0 e se

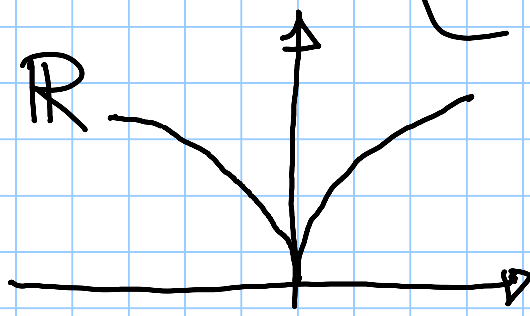
$$\lim_{x \to x_0^\pm} \frac{f(x)-f(x_0)}{x-x_0} = \pm\infty$$

o se

$$\lim_{x \to x_0^\pm} \frac{f(x)-f(x_0)}{x-x_0} = \mp\infty$$

x_0 si dice PUNTO di CUSPIDE

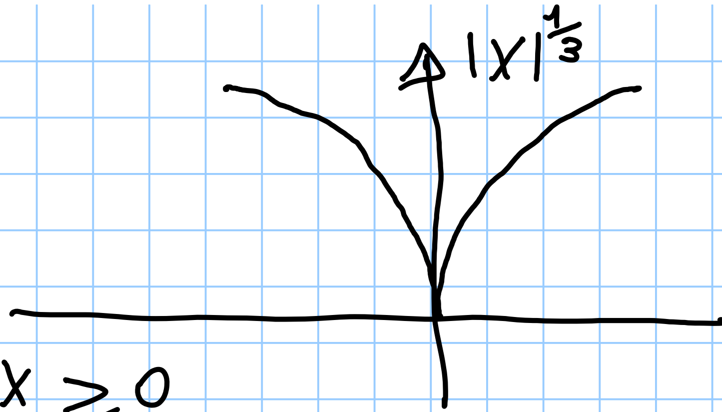
$f(x) = \sqrt{|x|}$, $x \in \mathbb{R}$



$$\lim_{x \to 0^\pm} \frac{\sqrt{|x|}}{x} = \pm\infty$$

$$\frac{\sqrt{|x|}}{x} = \begin{cases} \dfrac{\sqrt{x}}{x} = \dfrac{1}{\sqrt{x}}, & x>0 \\ \dfrac{\sqrt{|x|}}{-|x|} = -\dfrac{\sqrt{|x|}}{\sqrt{|x|}\sqrt{|x|}} = -\dfrac{1}{\sqrt{|x|}}, & x<0 \end{cases}$$

$$f(x) = |x|^{1/3}$$



$$f(x) = \begin{cases} x^{1/3}, & x \geq 0 \\ (-x)^{1/3}, & x < 0 \end{cases}$$

$$f'(x) = \begin{cases} \frac{1}{3} x^{-2/3}, & x > 0 \\ \frac{1}{3}(-x)^{-2/3}(-1), & x < 0 \end{cases}$$

$$\lim_{x \to 0^\pm} f'(x) = \pm\infty = \lim_{x \to 0^\pm} \frac{f(x) - f(0)}{x - 0}$$