$$\boxed{x^2 \cdot o(x) = o(x^3)}_{x\to 0} \; ; \quad \frac{\cancel{x^2} \cdot o(x)}{x^{\cancel{3}}} = \frac{o(x)}{x} = o(1)_{x\to 0}$$

$x \to 0$

$$\boxed{x^{-\frac{1}{3}} \cdot o(x) = o\left(x^{\frac{2}{3}}\right)}_{x\to 0^+} \; ; \quad \frac{x^{-\frac{1}{3}} \cdot o(x)}{x^{2/3}} = \frac{o(x)}{x} = o(1)$$

$$g(x) \cdot o(f(x)) = o(g(x) \cdot f(x))$$

(Se $\exists \lim\limits_{x\to x_0} g$ e se $f(x) = o(1)$, $x \to x_0$
o se $f(x) \to +\infty$, $x \to x_0$)

$$\left(\frac{1}{x}\right) \cdot o(x) \underset{x\to 0}{=} o\left(\frac{x}{x}\right) = o(1)$$

$$\frac{1}{\sqrt{x}}\, o(x) = o(\sqrt{x}), \quad x \to 0$$

$$\frac{1}{x^2} \cdot o(x) = o\left(\frac{1}{x}\right), \quad x \to 0$$

$$o(x+x^2) = o(x); \quad x^2 = o(x)$$
$$x \to 0$$

$$\frac{o(x+x^2)}{x} = \frac{o(x+o(x))}{x} = \frac{o(x(1+o(1)))}{x}$$

$$= \frac{o(x)}{x} = o(1), \quad x \to 0$$

Se $f(x)$ è infinitesima $x \to x_0$
o infinito $x \to x_0$,

$$o(f(x) + o(f(x))) = o(f(x))$$

$$\lim_{x \to \pi} \frac{\log\left(e + \sin(x)\right) + \cos(x)}{\cos\left(\frac{x}{2}\right)} =$$

$$x = \pi + y \qquad y \to 0$$

$$\sin(x) = \sin(\pi + y) =$$

$$= \sin\pi \cos(y) + \sin y \cos(\pi) =$$

$$= -\sin(y)$$

$$\cos(x) = \cos(\pi + y) =$$

$$= \cos(\pi)\cos(y) - \sin(\pi)\sin(y)$$

$$= -\cos(y)$$

$$= \lim_{y \to 0} \frac{\log\left(e - \sin(y)\right) - \cos(y)}{\cos\left(\frac{y + \pi}{2}\right)} =$$

$$\cos\left(\frac{y}{2}+\frac{\pi}{2}\right) = \cos\left(\frac{y}{2}\right)\cos\left(\frac{\pi}{2}\right) - \sin\left(\frac{y}{2}\right)\sin\left(\frac{\pi}{2}\right) = -\sin\left(\frac{y}{2}\right)$$

$$= \lim_{y\to 0} \frac{\log(e-\sin(y)) - \cos(y)}{-\sin\left(\frac{y}{2}\right)} =$$

$$\log(e-\sin(y)) \underset{y\to 0}{=} \log(e - y + o(y))$$

$$= \log e + \log\Big(1 \underbrace{- \frac{y}{e} + o(y)}_{x}\Big) =$$

$$= 1 \overbrace{- \frac{y}{e} + o(y)}^{x} + \underset{o(x)}{o\left(\frac{y}{e} + o(y)\right)}$$

$$= 1 - \frac{y}{e} + o(y)$$

$$\cos(y) = 1 - \frac{y^2}{2} + o(y^2)$$

$$\lim_{y\to 0} \ldots\ldots = \lim_{y\to 0} \frac{\cancel{1} - \frac{y}{e} + \boxed{o(y)} - \cancel{1} + \frac{y^2}{2} + o(y^2)}{-\frac{y}{2} + o(y)} =$$

$$= \lim_{y\to 0} \frac{-y/e + \frac{y^2}{2} + o(y)}{-\frac{y}{2} + o(y)} =$$

$$= \lim_{y\to 0} \frac{-y/e + o(y)}{-y/2 + o(y)} = \lim_{y\to 0} \frac{y/e}{y/2}(1+o(1)) = 2/e.$$

Ex $$\lim_{x\to 1} \frac{\log(e+\sin(\pi x)) + \cos(\pi x)}{\cos(\pi \frac{x}{2})}$$

$$\lim_{x\to 0^+} \frac{Tg(x) - \sin(x)}{\sqrt{x}} =$$

$$\sin(x) = x + o(x) \quad , x \longrightarrow 0$$

$$Tg(x) = x + o(x) \quad , x \longrightarrow 0$$

$$\lim_{x\to 0^+} \frac{\cancel{x} + o(x) - \cancel{x} + o(x)}{\sqrt{x}} = \lim_{x\to 0^+} \frac{o(x)}{\sqrt{x}} = 0$$

$$\frac{o(x)}{x} \xrightarrow[x\to 0]{} 0 \quad ; \quad \frac{o(x)}{\sqrt{x}} = \frac{o(x)}{x} \frac{x}{\sqrt{x}} = o(1)\sqrt{x} \xrightarrow[x\to 0]{} 0$$

$$\frac{o(x)}{\sqrt{x}} = \frac{o\left(\sqrt{x} \cdot \sqrt{x}\right)}{\sqrt{x}} = \cancel{\sqrt{x}}\, \frac{o\left(\sqrt{x}\right)}{\cancel{\sqrt{x}}} = o\left(\sqrt{x}\right) \xrightarrow[x\to 0]{} 0$$

$$\lim_{x\to 0} \frac{Tg(x)-\sin(x)}{x^3} =$$

$$= \lim_{x\to 0} \frac{x+o(x) - x+o(x)}{x^3} = \lim_{x\to 0} \frac{o(x)}{x^3} ?$$

$$\frac{o(x)}{x^3} = \frac{o(x)}{x} \cdot \frac{x}{x^3} = o(1) \cdot \frac{1}{x^2} \to 0 \cdot +\infty$$

F.I.

$o(x)$ E' PER DEFINIZIONE UN INFINITESIMO di ORDINE SUPERIORE a x, MA NON SAPPIAMO SE è ANCHE di ORDINE SUPERIORE A x^3!

ESEMPIO: $x^2 = o(x)$ ma non è $o(x^3)$; $x^4 = o(x)$; $x^4 = o(x^3)$

$$\lim_{x \to 0} \frac{Tg(x) - Sin(x)}{x^3} =$$

$$\lim_{x \to 0} \frac{Sin(x)}{x} \frac{1 - cos(x)}{x^2} \frac{1}{cos(x)} = \frac{1}{2}$$

$$\Rightarrow Tg(x) - Sin(x) \underset{x \to 0}{=} \frac{1}{2} x^3 + o(x^3)$$

$$\lim_{x\to 0} \frac{(2^x - 1)^2}{\sin(x)\log(1+x)} =$$

$$= \lim_{x\to 0} \frac{(e^{x\log 2} - 1)^2}{\sin(x)\log(1+x)} =$$

$$\frac{e^y - 1}{y} \to 1, \; y \to 0 \implies e^y = 1 + y + o(y)$$
$$y \to 0$$

$$y = x\log(2)$$

$$\sin x = x + o(x), \quad x \to 0$$

$$\log(1+x) = x + o(x), \quad x \to 0$$

$$= \lim_{x\to 0} \frac{(\cancel{1} + x\log(2) + o(x) - \cancel{1})^2}{(x + o(x))(x + o(x))} =$$

$$= \lim_{x\to 0} \frac{(x\log(2))^2\,(1+o(1))}{x(1+o(1))\,x(1+o(1))} =$$

$$= (\log(2))^2$$

$$x+o(x) = x\left(1+\frac{o(x)}{x}\right) = x(1+o(1))$$

$$\lim_{x\to 3} \frac{e^x - e^3}{\log(x^3-26)} =$$

$$\log(1+y) = y+o(y), \quad y\to 0$$

$$\log(x^3-26) = \log(1+(x^3-27))$$

$$\log(1+(x^3-27)) = (x^3-27)+o(x^3-27)$$

$$= (x^3 - 27)(1 + o(1))$$

$$= \lim_{x \to 3} \frac{e^x - e^3}{(x^3 - 27)(1 + o(1))} =$$

$$e^x - e^3 = e^3 (e^{x-3} - 1) =$$

$$= e^3 (\cancel{1} + (x-3) + o(x-3) - \cancel{1}) =$$

$$= e^3 (x-3)(1 + o(1))$$

$$= \lim_{x \to 3} \frac{e^3 (x-3)}{(x^3 - 27)} \frac{(1 + o(1))}{(1 + o(1))} =$$

$$= \lim_{x \to 3} e^3 \frac{x-3}{x^3-27} =$$

$$y = \frac{x}{3}$$

$$x = 3y, \quad y \to 1$$

$$= \lim_{y\to 1} e^3 \frac{3y-3}{27y^3-27} = \lim_{y\to 1} \frac{e^3}{9} \frac{y-1}{y^3-1} =$$

$$= \frac{e^3}{9} \cdot \frac{1}{3} = \frac{e^3}{27}$$

$$\boxed{\lim_{x\to 1} \frac{x^\alpha - 1}{x^\beta - 1} = \frac{\alpha}{\beta} \quad , \beta \neq 0}$$

$$\lim_{x \to 0^+} \frac{\sin(x^3+x^5) - \left(\log(1+x^{\frac{1}{3}})\right)^{\pi}}{x^{\frac{4}{3}}}.$$

$$\sin(x^3+x^5) = x^3+x^5+o(x^3+x^5)$$

$$\sin(y) = y+o(y) \; ; \; x^5 = o(x^3)$$

$$= x^3+o(x^3)+o(x^3+o(x^3))$$

$$= x^3+o(x^3)+o(x^3) = x^3+o(x^3), \quad x \to 0^+$$

$$\left(\log(1+x^{\frac{1}{3}})\right)^{\pi} = \left(x^{\frac{1}{3}}+o(x^{\frac{1}{3}})\right)^{\pi}$$

$$= x^{\frac{\pi}{3}}\left(1+o(1)\right)^{\pi} = x^{\frac{\pi}{3}}(1+o(1))$$

$$\left\{(1+o(1))^{\pi} = 1+\pi \cdot o(1)+o(1) = 1+o(1)\right\}$$

$$(1+y)^{\alpha} \underset{y \to 0}{=} 1+\alpha y+o(y) = 1+o(1)$$

$$\lim_{x\to 0^+} \frac{x^3 + o(x^3) - x^{\frac{\pi}{3}} + o(x^{\frac{\pi}{3}})}{x^{\frac{4}{3}}} =$$

$x^3 = o(x^{\frac{\pi}{3}})$ $\qquad$ $o(x^3) = o(x^{\frac{\pi}{3}})$

$1 < \frac{\pi}{3} < 2$

$$= \lim_{x\to 0^+} \frac{x^3 - x^{\frac{\pi}{3}} + o(x^{\frac{\pi}{3}})}{x^{\frac{4}{3}}} =$$

$$= \lim_{x\to 0^+} \frac{-x^{\frac{\pi}{3}} + o(x^{\frac{\pi}{3}})}{x^{4/3}} =$$

$$= \lim_{x\to 0^+} -\frac{1 + o(1)}{x^{\frac{4-\pi}{3}}} = -\frac{1}{0^+} = -\infty$$

Ex

$$\lim_{x\to 0^-} \frac{x}{\sqrt{2 - e^{x + x^{1/3}}} - 1} \quad (= 0^-)$$

ASINTOTO OBLIQUO

$$f(x) = \frac{x}{3} 2^{\frac{1}{x}} - 1$$

$$\lim_{x \to -\infty} f(x) = (-\infty) \cdot 2^{0} - 1 = -\infty$$

$$f(x) = \frac{x}{3} 2^{\frac{1}{x}} - 1 = \frac{x}{3} e^{\frac{\log 2}{x}} - 1$$

$$e^{y} = 1 + y + o(y) \quad y \to 0$$

$$= \frac{x}{3}\left(1 + \frac{\log 2}{x} + o\left(\frac{1}{x}\right)\right) - 1$$

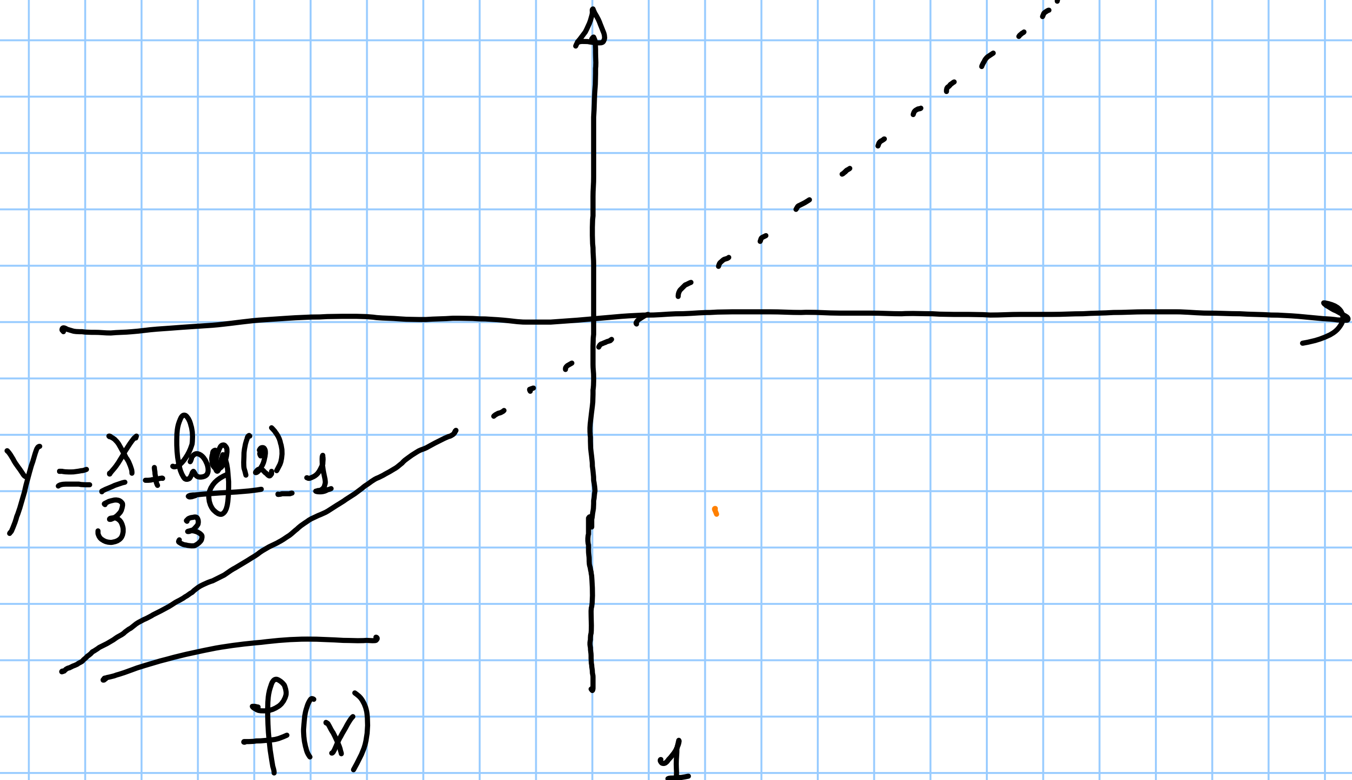
$$= \frac{x}{3} + \frac{\log 2}{3} + o(1) - 1$$

$$f(x) = \frac{x}{3} + \frac{\log 2}{3} - 1 + o(1)$$

$$r : \left\{ (x, y) \in \mathbb{R}^2 : y = \frac{x}{3} + \frac{\log 2}{3} - 1 \right\}$$

ASINTOTO OBLIQUO di $f(x)$ per $x \to -\infty$

$$f(x) = \frac{x}{3} 2^{1/x} - 1$$



Ex $f(x) = \frac{x}{3} 2^{1/x} - 1, \quad x \longrightarrow +\infty$

....

DEFINIZIONE [ASINTOTO OBLIQUO]

Se $\exists\, m \in \mathbb{R} / \{0\}$ e $q \in \mathbb{R}$:

$$f(x) = mx + q + o(1) \,, \quad x \to +\infty$$

$$\left(f(x) = mx + q + o(1) \,, \quad x \to (-\infty) \right)$$

ALLORA f ha un ASINTOTO OBLIQUO per $x \to +\infty$ $(-\infty)$

$$r = \left\{ (x,y) \in \mathbb{R}^2 : y = mx + q \right\}$$

$$f(x) = mx + q + o(1) \,, \quad x \to +\infty$$

$$\lim_{x \to +\infty} f(x) - mx - q = 0$$

$$\Downarrow$$

$$\lim_{x \to +\infty} \frac{f(x) - mx - q}{x} = 0$$

$$\lim_{x \to +\infty} \frac{f(x)}{x} - m - \underbrace{\frac{q}{x}}_{\to 0} = 0$$

DEFINIZIONE EQUIVALENTE

(i) $\lim_{\substack{x \to +\infty \\ (-\infty)}} \frac{f(x)}{x} = m$

$\exists m \in \mathbb{R} \setminus \{0\}$

(ii) $\lim_{\substack{x \to +\infty \\ (-\infty)}} f(x) - mx = q$

$\exists q \in \mathbb{R}$

Allora la retta $\{y = mx + q\}$

è un ASINTOTO OBLIQUO per

$x \to \begin{matrix} +\infty \\ (-\infty) \end{matrix}$

ATTENZIONE:

L'ESISTENZA di $m \neq 0$ $\not\Rightarrow$
l'ESISTENZA di q.

$$f(x) = x + \log(x)$$

$$\exists\ m = \lim_{x \to +\infty} \frac{f(x)}{x} = \lim_{x \to +\infty} \frac{x + \log(x)}{x} = 1$$

$$\nexists\ q = \lim_{x \to +\infty} f(x) - mx = \lim_{x \to +\infty} f(x) - x$$

$$= \lim_{x \to +\infty} \log(x) = +\infty$$

Ex $f(x) = x + \sqrt{x}$ .....

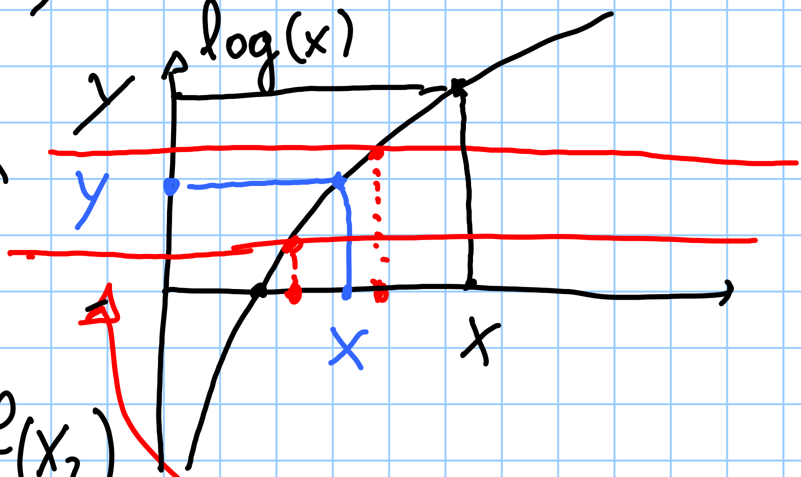
LA FUNZIONE INVERSA

$f : X \longrightarrow Y$, INIETTIVA E SURIETTIVA. La funzione inversa di f è l'unica funzione $g : Y \rightarrow X$:

$$f(g(y)) = y \quad \forall y \in Y$$

$$g(f(x)) = x \quad \forall x \in X$$

$f(x) = \log(x)$, $x \in X = (0, +\infty)$



$f : X \to \mathrm{Im}(f) = \mathbb{R}$

f E' INIETTIVA

$x_1 \neq x_2 \Longrightarrow f(x_1) \neq f(x_2)$

PER OGNI VALORE ASSUNTO da f SI HA UNA SOLA CONTROIMMAGINE

$\forall y \in \mathrm{Im}(f)$ $\exists$ un unico $x \in X$: $y = f(x)$

LA FUNZIONE CHE AD OGNI $y \in \mathrm{Im}(f)$ ASSOCIA L'UNICA CONTROIMMAGINE di y in X

E' LA FUNZIONE INVERSA di f (f^{-1})

$y = \log x \quad , \quad e^{y} = x \longrightarrow$

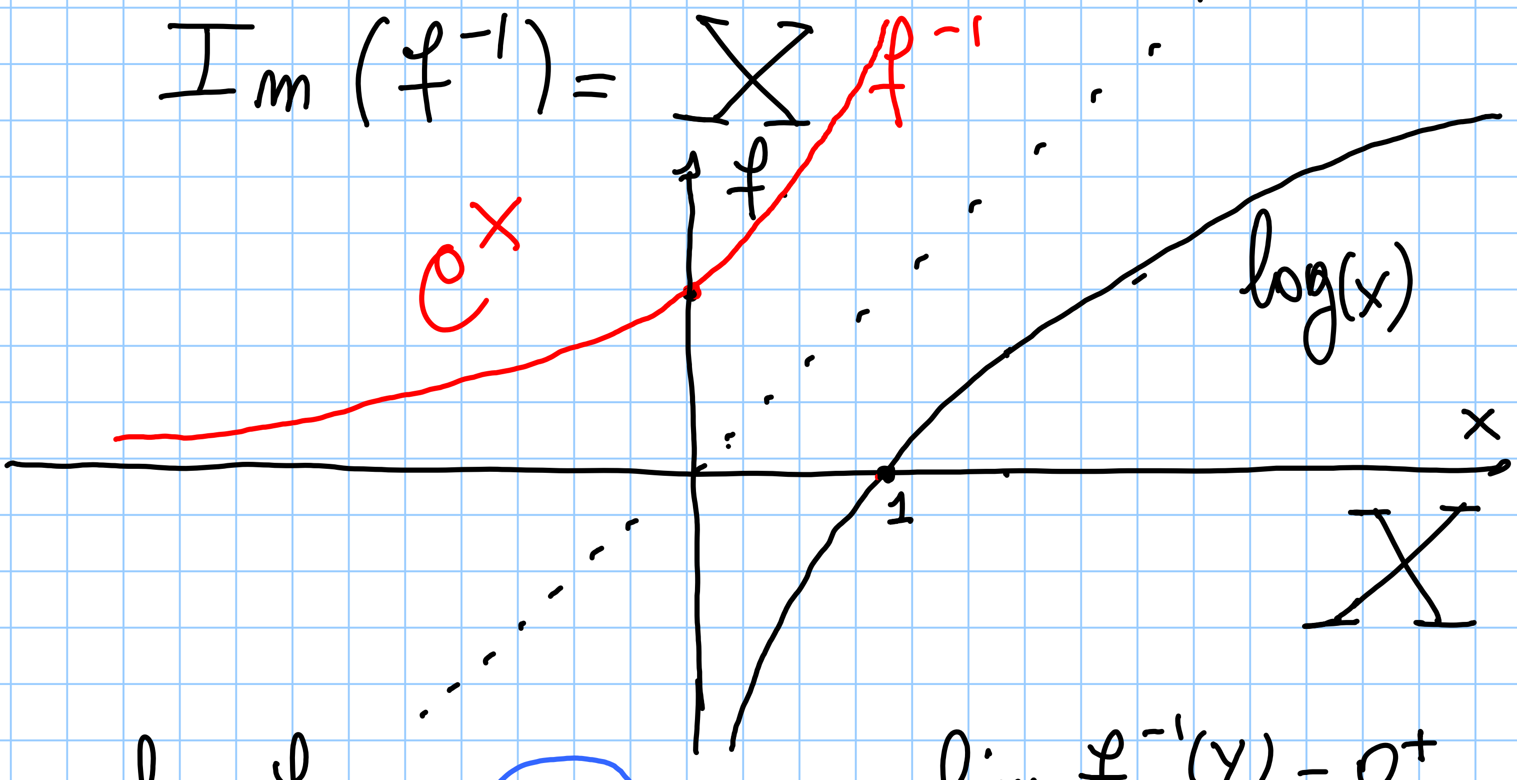
$x = e^{y} \quad , \quad f^{-1}(y) = e^{y}$

$f^{-1} : \mathrm{Im}(f) \longrightarrow X$

$f^{-1} : \mathbb{R} \longrightarrow (0, +\infty)$

dominio di $f^{-1} = \mathrm{Im}(f)$

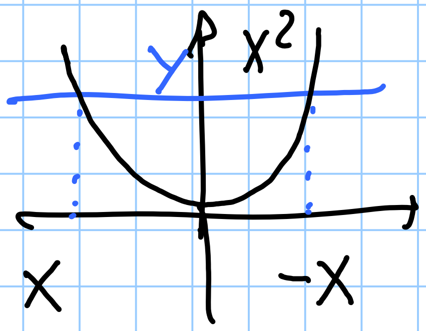
$\mathrm{Im}(f^{-1}) = X$



$\lim_{x \to 0^+} f(x) = -\infty$

$\lim_{y \to -\infty} f^{-1}(y) = 0^+$

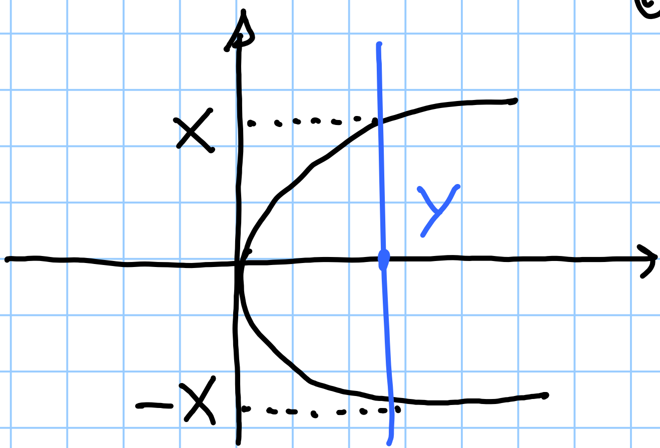
$\left(\lim_{f(x) \to 0} x = 0 \right)$



$f(x)=x^2,\ x\in\mathbb{R}$

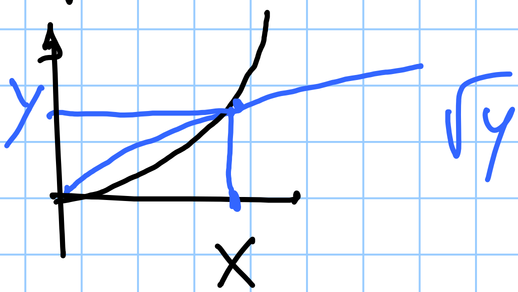
NON E' INIETTIVA

quindi non $\boxed{\exists \text{ L'INVERSA}}$



QUESTO NON E' IL GRAFICO di UNA FUNZIONE

$f(x)=x^2,\ x\in[0,+\infty)$



E' INIETTIVA in $[0,+\infty)$

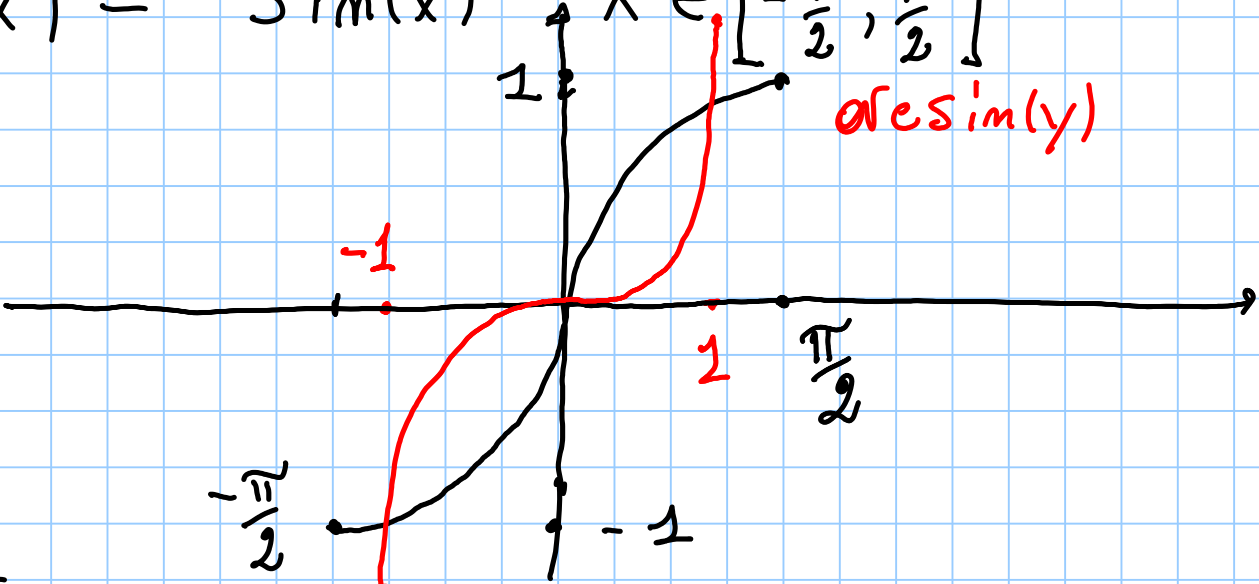
$\Downarrow$

$\exists$ L'INVERSA

$y=x^2,\quad x=\sqrt{y}$

$f^{-1}(y)=\sqrt{y},\quad y\in[0,+\infty)$

$f(x) = \sin(x)$, $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$



$X = [-\frac{\pi}{2}, \frac{\pi}{2}]$, $Im(f) = [-1, 1]$

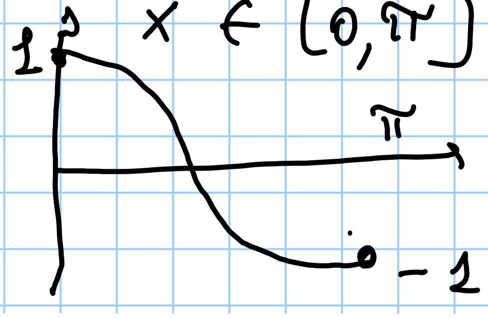
$f^{-1}(y) = \arcsin(y)$, $y \in [-1, 1]$

$Im(f^{-1}) = [-\frac{\pi}{2}, \frac{\pi}{2}]$

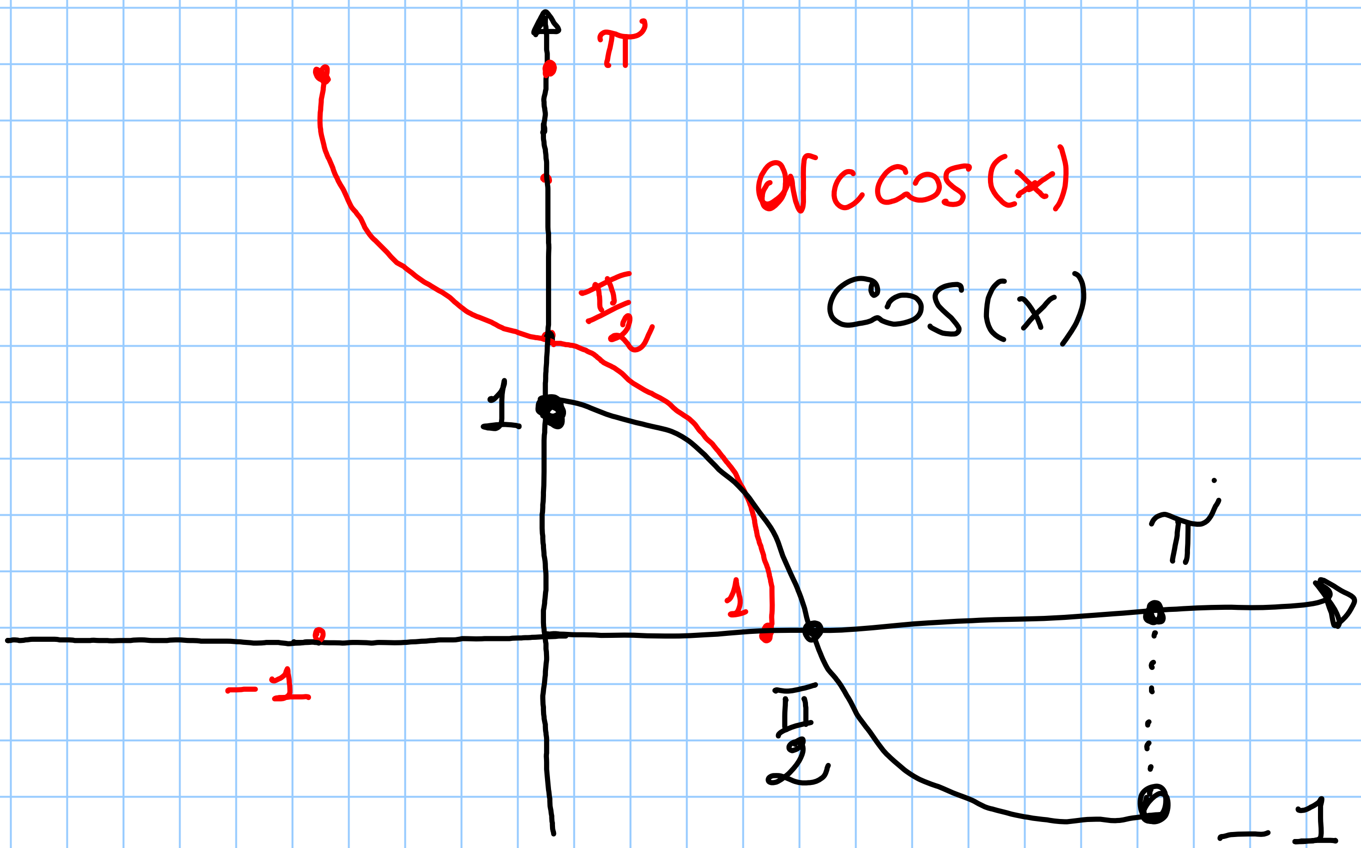
$\arcsin(1) = \frac{\pi}{2}$ $\quad\quad$ $\sin(\frac{\pi}{2}) = 1$

$\arcsin(-1) = -\frac{\pi}{2}$ $\quad\quad$ $\sin(-\frac{\pi}{2}) = -1$

$f(x) = \cos(x)$ $\quad$ $x \in [0, \pi]$



$Im(f) = [-1, 1]$



$$f^{-1}(y) = \arccos(y), \quad y \in [-1,1]$$

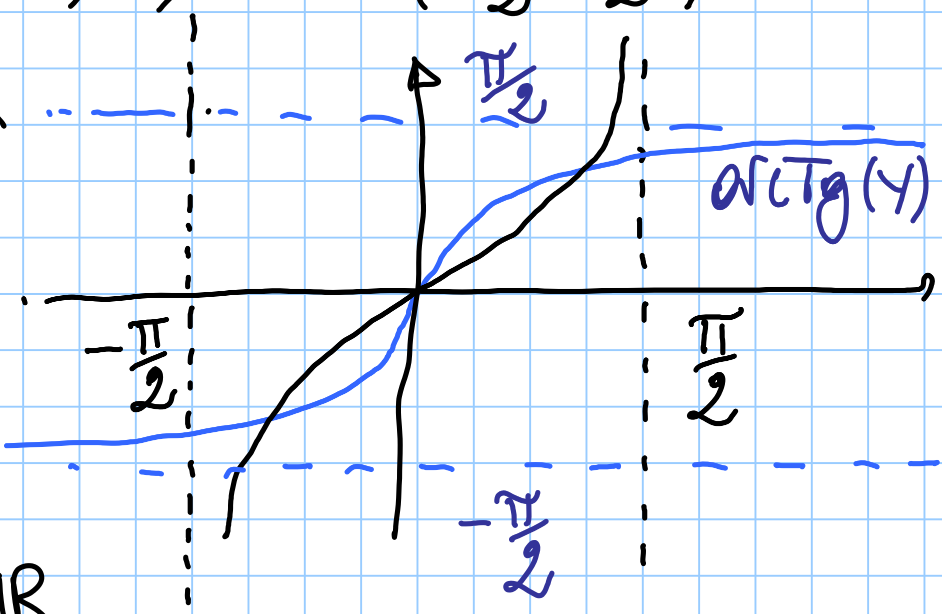
$$\mathrm{Im}(f^{-1}) = [0,\pi]$$

$\arccos(1) = 0$ $\qquad$ $\cos(0) = 1$

$\arccos(-1) = \pi$ $\qquad$ $\cos(\pi) = -1$

$$f(x) = Tg(x), \quad x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$$Im(f) = \mathbb{R}$$



$$f^{-1}(y) = arcTg(y)$$

$$Im(f^{-1}) \overset{y \in \mathbb{R}}{=} \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$$\lim_{y \to +\infty} arcTg(y) = \frac{\pi}{2}^{-} \qquad \lim_{x \to \left(\frac{\pi}{2}\right)^{-}} Tg(x) = +\infty$$

$$\lim_{y \to -\infty} arcTg(y) = \left(-\frac{\pi}{2}\right)^{+} \qquad \lim_{x \to \left(-\frac{\pi}{2}\right)^{+}} Tg(x) = -\infty$$

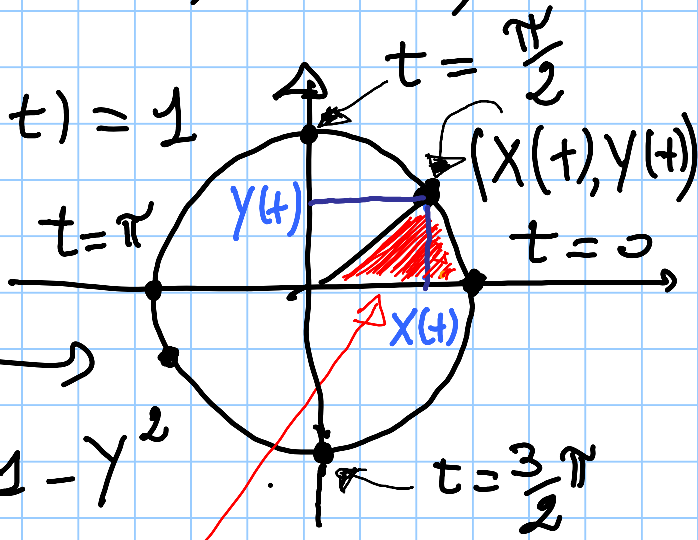
FUNZIONI IPERBOLICHE E LORO INVERSE

$$\begin{cases} x(t) = \cos(t) \\ y(t) = \sin(t) \end{cases} \qquad t \in [0, 2\pi)$$

$$(x(t), y(t)) = (\cos(t), \sin(t))$$

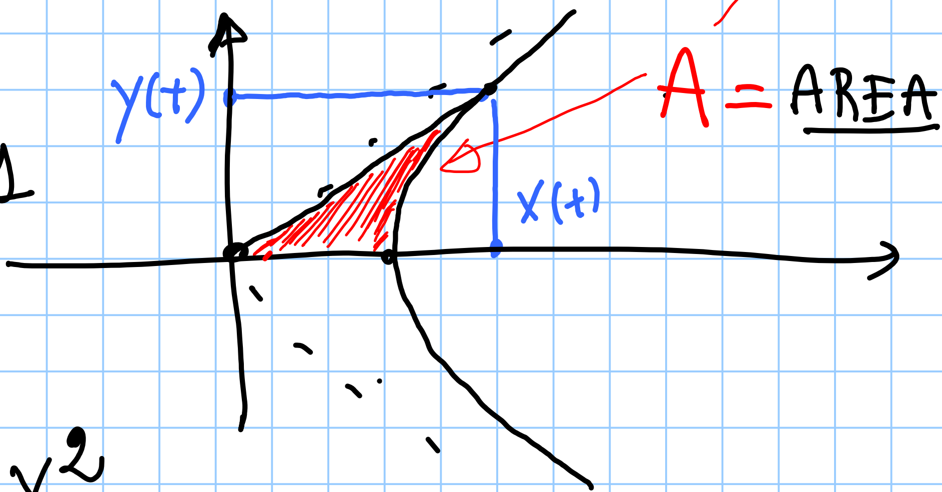
$$\cos^2(t) + \sin^2(t) = 1$$

$$\boxed{x^2 + y^2 = 1}$$



$$x^2 = 1 - y^2$$

$$\begin{cases} x^2 - y^2 = 1 \\ \boxed{x > 0} \end{cases}$$



$$\boxed{t = 2A}$$

$$x^2 = 1 + y^2$$

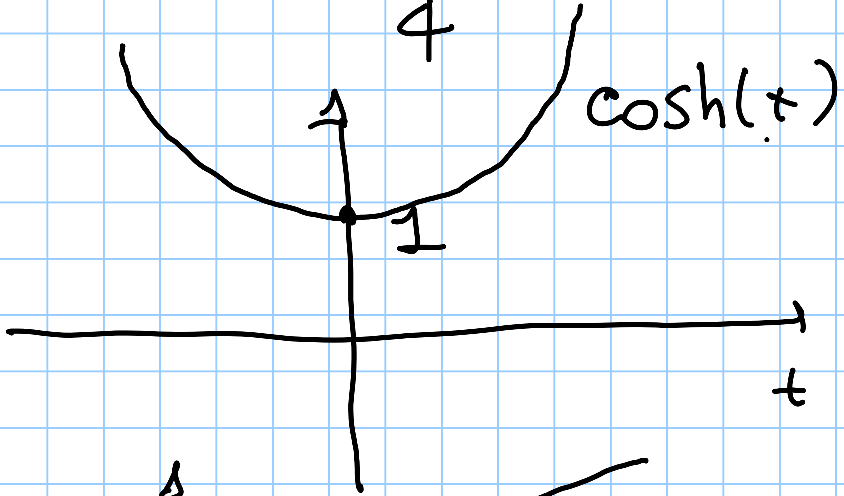
$$x(t) = \frac{e^t + e^{-t}}{2} = \cosh(t)$$

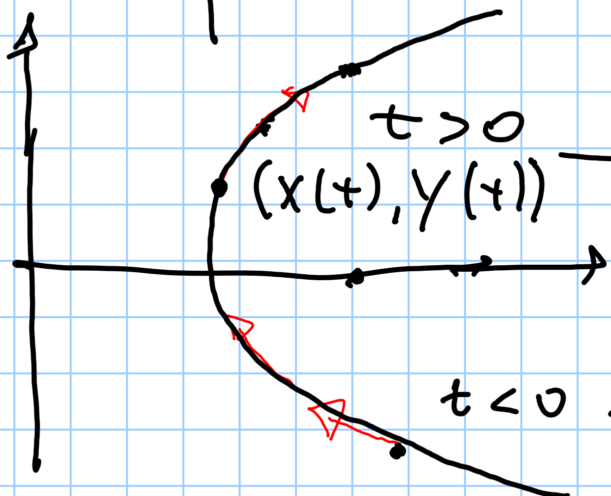
$$y(t) = \frac{e^t - e^{-t}}{2} = \sinh(t)$$

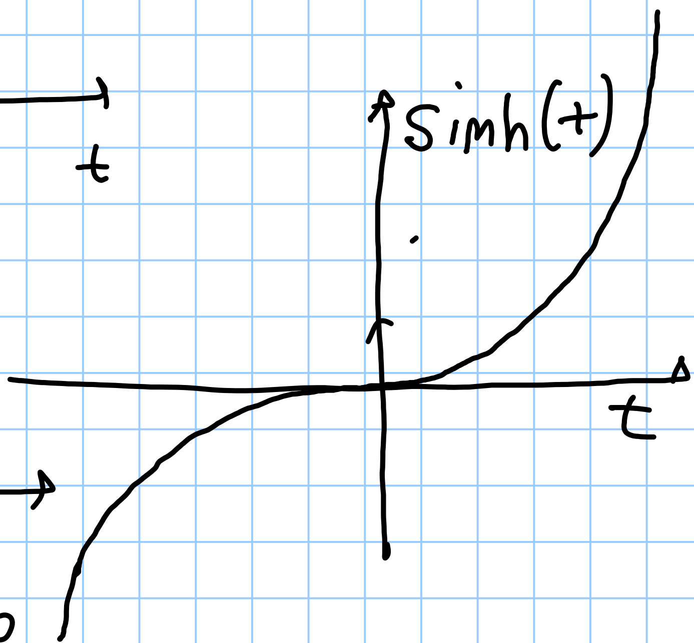
$(x(t), y(t))$, $x^2(t) - y^2(t) = 1$

$$\left(\frac{e^{t} + e^{-t}}{2}\right)^2 - \left(\frac{e^{t} - e^{-t}}{2}\right)^2 =$$

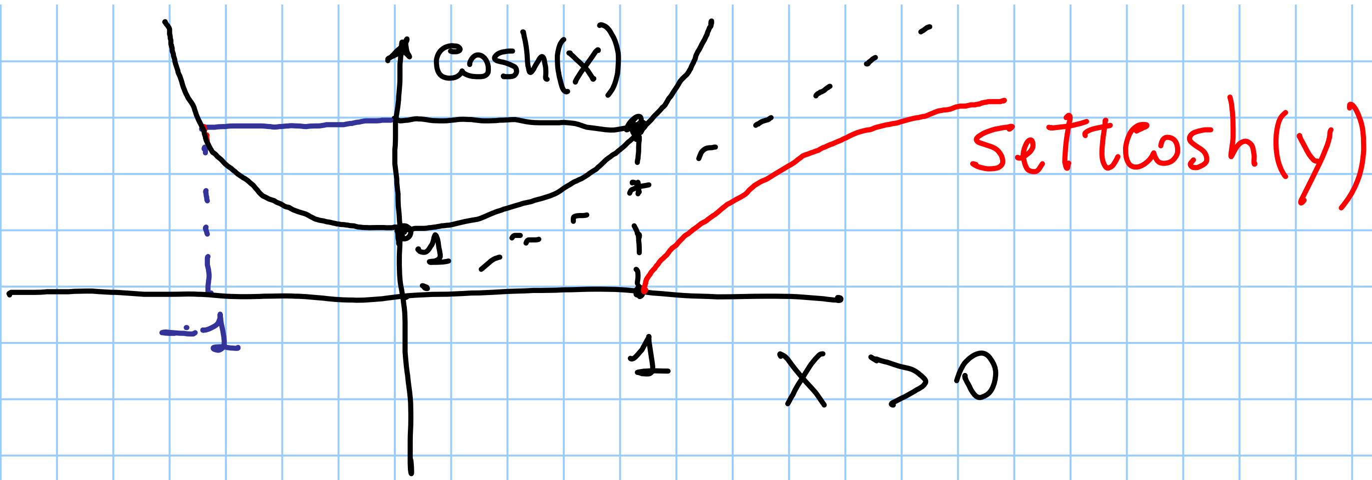
$$\frac{\cancel{e^{2t}} + 2 + \cancel{e^{-2t}}}{4} - \frac{\cancel{e^{2t}} - 2 + \cancel{e^{-2t}}}{4} = 1$$







$$(x(t), y(t)) = (\cosh(t), \sinh(t))$$



$x > 0$

$$f(x) = \cosh(x) \quad , \quad x \in [0, +\infty)$$

$$Im(f) = [1, +\infty)$$

È INIETTIVA

$$y = \cosh(x) = \frac{e^x + e^{-x}}{2}$$

$$e^x + e^{-x} = 2y$$

$$e^{2x} - 2e^x y + 1 = 0$$

$$e^x = z \quad , \quad z^2 - 2zy + 1 = 0$$

$$z_\pm = y \pm \sqrt{y^2 - 1}$$

$$e^{x_\pm} = y \pm \sqrt{y^2-1}$$

$$\text{Ex} \quad x_\pm = \log\left(y \pm \sqrt{y^2-1}\right)$$

$$x_-(y) < 0$$

$$0 < x_+(y) = \log\left(y + \sqrt{y^2-1}\right)$$

$$\boxed{\operatorname{settcosh}(y) = \log\left(y + \sqrt{y^2-1}\right)}$$

$$f^{-1}(y) : [1, +\infty) \longmapsto [0, +\infty)$$

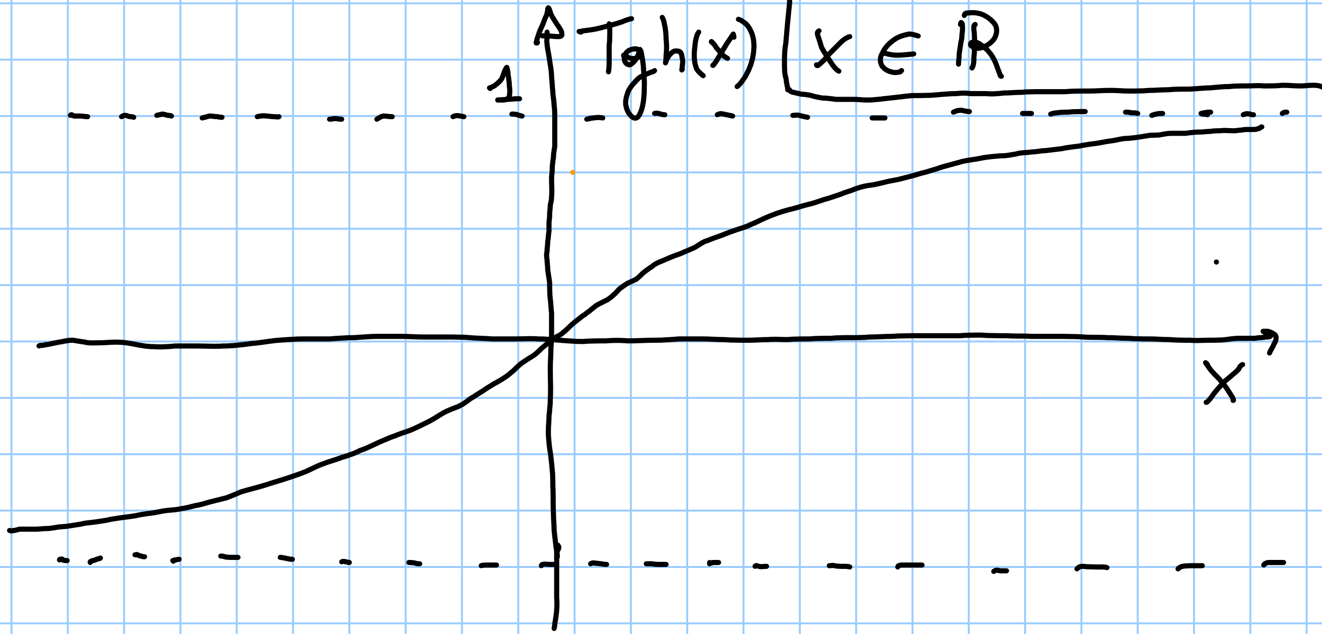
$$f(x) = \sinh(x), \quad x \in \mathbb{R}$$

$$f^{-1}(y) = \log\left(y + \sqrt{y^2+1}\right) =$$

$$= \operatorname{settsinh}(y)$$

$$\text{Ex} \quad \left(y - \sqrt{y^2+1}\right)?$$

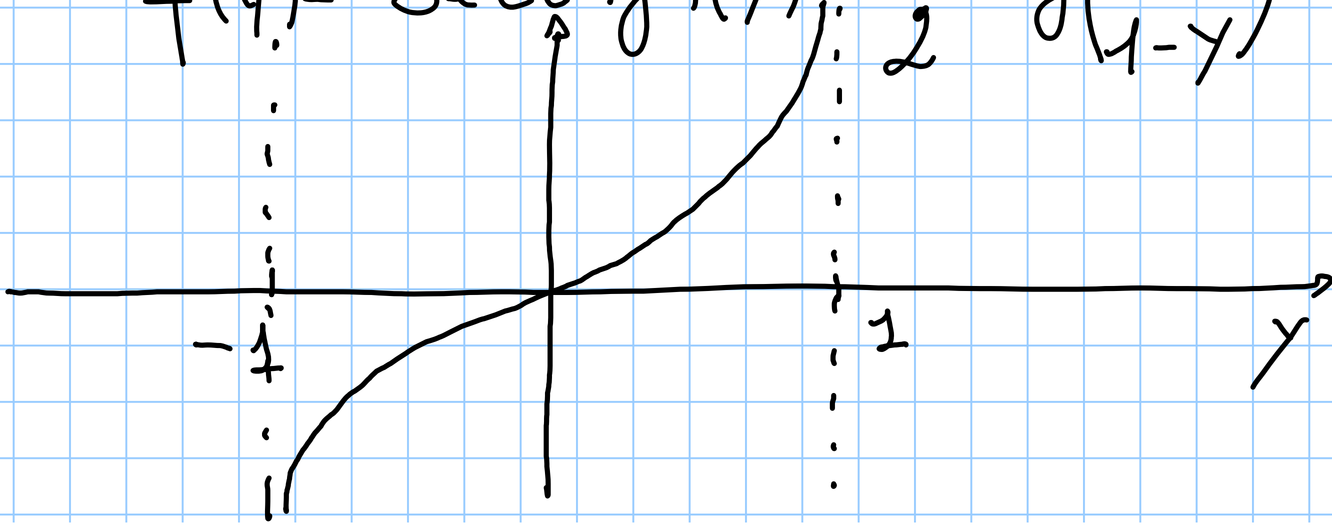
$$f(x) = Tgh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^{x} - e^{-x}}{e^{x} + e^{-x}}$$



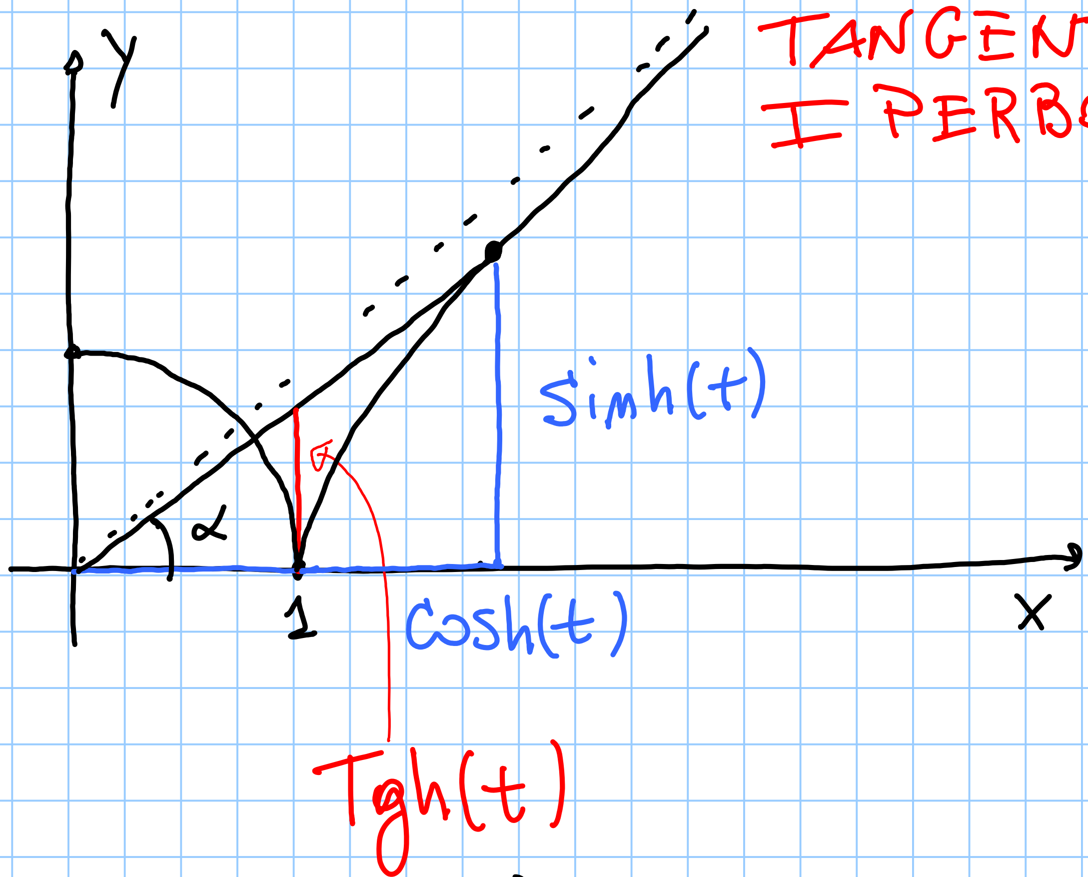
$$f : \mathbb{R} \longrightarrow (-1, 1)$$

$$f^{-1} : (-1, 1) \longrightarrow \mathbb{R}$$

$$f(y) = \mathrm{SettTgh}(y) = \frac{1}{2} \log\left(\frac{1+y}{1-y}\right)$$



SIGNIFICATO GEOMETRICO DELLA TANGENTE IPERBOLICA



$$Tgh(t) = \frac{\sinh(t)}{\cosh(t)} = Tg(\alpha)$$