$$\lim_{x \to x_0} f(x) = L \qquad f : X \longrightarrow \mathbb{R}$$

Definizione di) : Se

$$\lim_{n \to +\infty} f(a_n) = L \quad \forall \{a_n\} \subset X \setminus \{x_0\} :$$

$$a_n \longrightarrow x_0 \;,\; n \longrightarrow +\infty$$

$$\boxed{a_n \neq x_0}$$

f in generale non è DEFINITA in x_0 !

TEOREMA

$$\lim_{x \to x_0} f(x) = L \iff$$

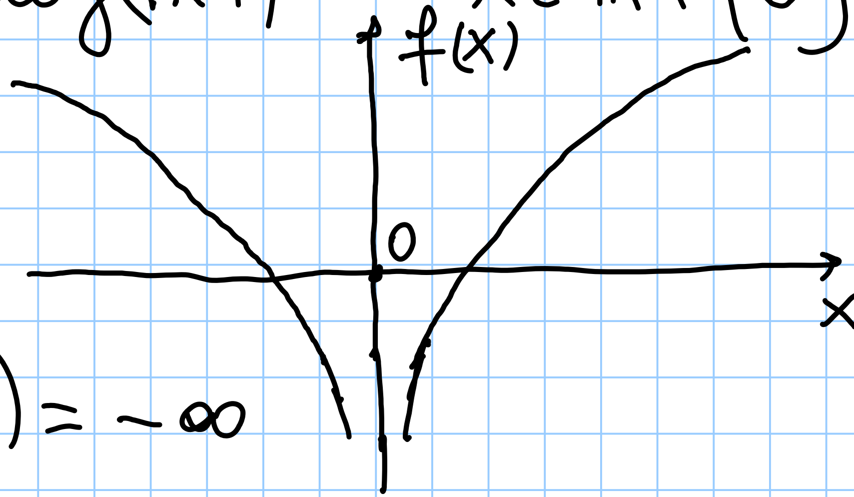
$$\forall V_L \;\; \exists U_{x_0} : f(x) \in V_L ,$$

intorno di L

intorno di x_0

$$\forall x \in \{U_{x_0} \cap X\} \setminus \{x_0\}$$

$$f(x) = \log(|x|) \qquad x \in \mathbb{R} \setminus \{0\}$$



$$\lim_{x \to 0} \log(|x|) = -\infty$$

ASINTOTO VERTICALE in $x_0 = 0$

[L' ASINTOTO è la RETTA

$$r = \{(x,y) \in \mathbb{R}^2 : x = 0\}$$]

$$\lim_{n \to +\infty} \log|a_n| = -\infty$$

$$\forall \{a_n\} \subset \mathbb{R} \setminus \{0\} :$$

$$a_n \to 0, \ n \to +\infty$$

$$b_n = |a_n| > 0, \ \forall n \in \mathbb{N}$$

$$\lim_{n \to +\infty} \log(b_n) = -\infty, \ b_n \xrightarrow[n \to +\infty]{} 0, \ b_n > 0 \ \forall n$$

$$\lim_{x \to 0} \log|x| = -\infty$$

$\forall$ intorno di $-\infty$ $(=L)$ $(-\infty, -M)$ $\quad$ $\exists$ intorno di $x_0 = 0$ $(-\delta_M, \delta_M)$

$\hookrightarrow \forall M > 0 \ \exists \delta_M > 0 : f(x) \in (-\infty, -M)$

$f(x) < -M,$

$\forall x \in (-\delta_M, \delta_M) \setminus \{0\}$

$\forall M > 0 \ \exists \delta_M > 0 : \log(|x|) < -M$
$\forall x \in (-\delta_M, \delta_M) \setminus \{0\}$

$$\log(|x|) < -M$$
$$\Updownarrow$$
$$|x| < e^{-M} =: \delta_M$$

$\boxed{E}$

ASINTOTO VERTICALE

$$\lim_{x \to x_0} f(x) = \pm\infty \quad r_0 = \{(x,y) \in \mathbb{R}^2 : x = x_0\}$$

$$\lim_{x \to 0} \frac{1}{|x|} = +\infty$$

$$\forall M > 0 \ \exists \ \delta_M > 0 : \ \frac{1}{|x|} > M$$

$$\forall x \in (-\delta_M, \delta_M) \setminus \{0\}$$

$$\frac{1}{|x|} > M \iff |x| < \frac{1}{M}$$

$$\delta_M = 1/M$$

$\blacksquare$

$$f(x) = 3 + \frac{1}{x^2+1}, \quad x \in \mathbb{R}$$

$$\lim_{x \to -\infty} f(x) = 3$$

$$\forall \varepsilon > 0 \quad \exists N_\varepsilon > 0 : |f(x) - 3| < \varepsilon,$$

$$\forall x \in (-\infty, -N_\varepsilon)$$

$$\left| 3 + \frac{1}{x^2+1} - 3 \right| < \varepsilon$$

$$\left| \frac{1}{x^2+1} \right| < \varepsilon$$

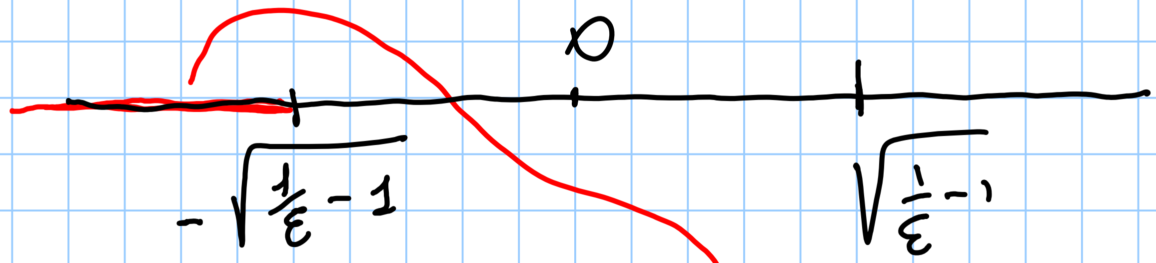
VERO $\forall x$ $\quad \boxed{-\varepsilon <} \frac{1}{x^2+1} < \varepsilon$

$$x^2 + 1 > \frac{1}{\varepsilon}$$

$$x^2 > \frac{1}{\varepsilon} - 1$$

$$x < -\sqrt{\frac{1}{\varepsilon} - 1} \quad , \quad x > \sqrt{\frac{1}{\varepsilon} - 1}$$

S.P.D.G. $\boxed{\varepsilon \leq 1 = \varepsilon_0}$



$$N_\varepsilon = \sqrt{\frac{1}{\varepsilon} - 1} \qquad x \in (-\infty, -N_\varepsilon)$$

ASINTOTO ORIZZONTALE

(per $x \longrightarrow -\infty$)

$$\lim_{x \to +\infty} f(x) = L \in \mathbb{R}$$

$$r = \{(x, y) \in \mathbb{R}^2 : y = L\}$$

$$\lim_{x \to -\infty} f(x) = L \in \mathbb{R}$$

Supponiamo che

$$f : X \longrightarrow \mathbb{R} \; ; \; X = (-\infty, 1) \cup (1, +\infty)$$

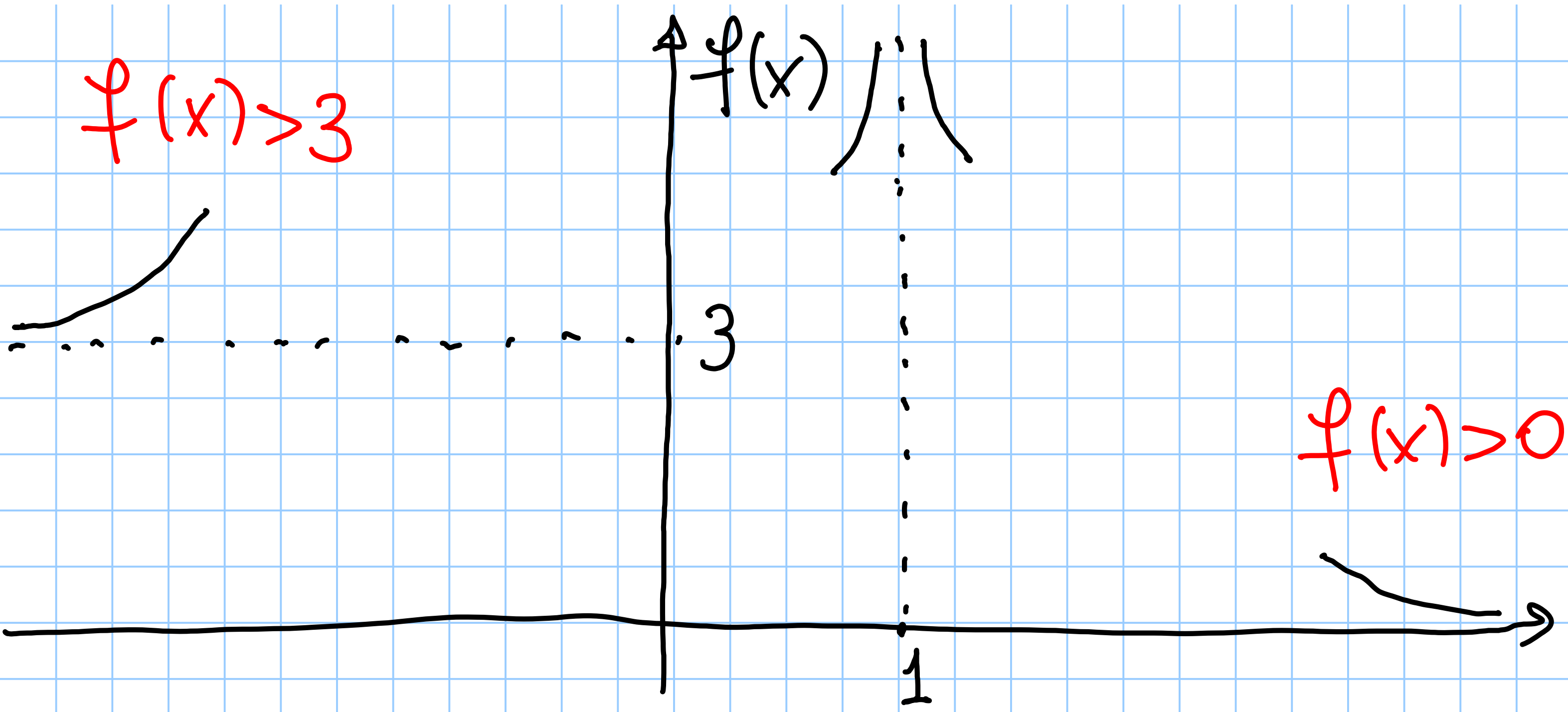
$$\lim_{x \to -\infty} f(x) = 3 \, ,$$

$$f(x) > 3 \quad \forall x < 0$$

$$\lim_{x \to 1} f(x) = +\infty$$

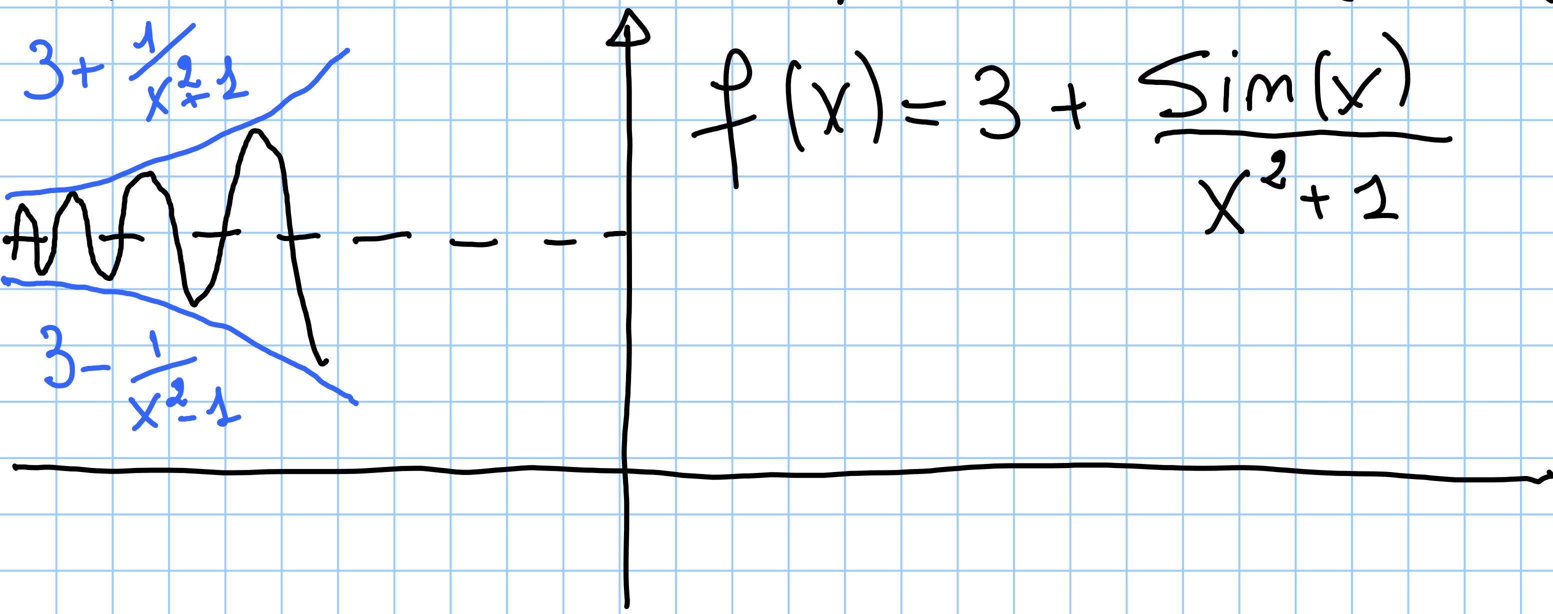
$$\lim_{x \to +\infty} f(x) = 0 \, ,$$

$$f(x) > 0 \, , \quad \forall x > 0$$



In generale f(x) può oscillare intorno all'asintoto orizzontale

$$f(x) = 3 + \frac{\sin(x)}{x^2+1}$$



$$f(x) = x^2 - 2 \quad , \quad x \in \mathbb{R}$$

$$\lim_{x \to \sqrt{2}} x^2 - 2 = 0$$

$$\lim_{x \to \sqrt{2}} x^2 - 2 = 0$$

$$\forall \varepsilon > 0 \ \exists \delta_\varepsilon > 0 : \ |x^2 - 2| < \varepsilon, \quad \forall x \in (\sqrt{2} - \delta_\varepsilon, \sqrt{2} + \delta_\varepsilon) \setminus \{\sqrt{2}\}$$

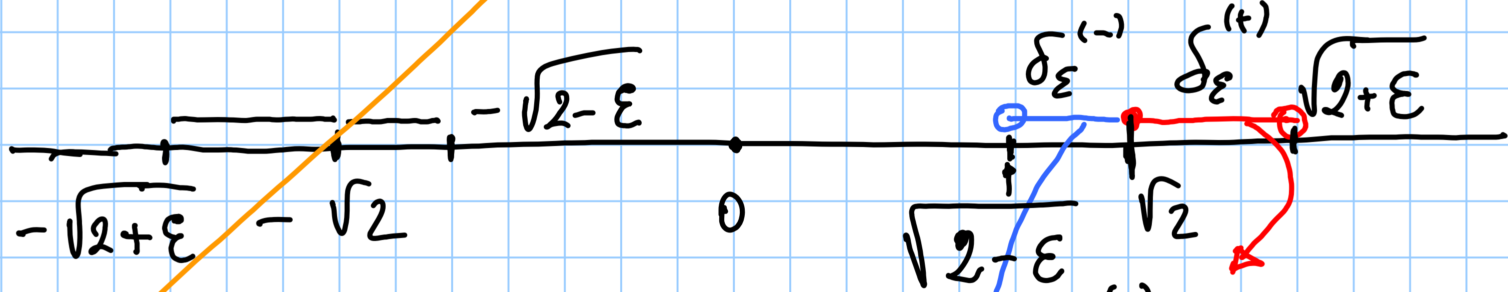
$$|x^2 - 2| < \varepsilon$$

$$\begin{cases} x^2 - 2 < \varepsilon \\ x^2 \geq 2 \end{cases} \cup \begin{cases} x^2 - 2 > -\varepsilon \\ x^2 < 2 \end{cases}$$

$$\begin{cases} x^2 < 2+\varepsilon \\ x^2 \geq 2 \end{cases} \cup \begin{cases} x^2 > 2-\varepsilon \\ x^2 < 2 \end{cases}$$

S.P.D.G. $\varepsilon \leq \varepsilon_0 = 2$

$$\begin{cases} -\sqrt{2+\varepsilon} < x < \sqrt{2+\varepsilon} \\ x \leq -\sqrt{2}\ ,\ x \geq \sqrt{2} \end{cases} \cup \begin{cases} x < -\sqrt{2-\varepsilon}, x > \sqrt{2-\varepsilon} \\ -\sqrt{2} < x < \sqrt{2} \end{cases}$$



$$\delta_\varepsilon^{(+)} = \sqrt{2+\varepsilon} - \sqrt{2}$$

$$\boxed{\delta_\varepsilon := \min\left\{\delta_\varepsilon^{(+)}, \delta_\varepsilon^{(-)}\right\}}$$

$$\delta_\varepsilon^{(-)} = \sqrt{2} - \sqrt{2-\varepsilon}$$

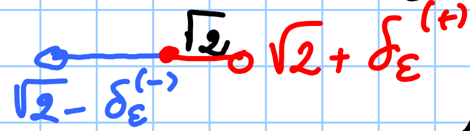
$\forall \varepsilon \in (0,2]$ ABBIAMO TROVATO

(*) $U_{\sqrt{2}} := \left(\sqrt{2-\varepsilon}, \sqrt{2+\varepsilon}\right) = \left(\sqrt{2} - \delta_\varepsilon^{(-)}, \sqrt{2} + \delta_\varepsilon^{(+)}\right)$

$: \ x^2 - 2 \in (-\varepsilon, \varepsilon),\ \forall x \in U_{\sqrt{2}} \setminus \{\sqrt{2}\}$

NON E' NECESSARIO

$$\delta_\varepsilon^{(-)} = \sqrt{2}-\sqrt{2-\varepsilon} > \sqrt{2+\varepsilon}-\sqrt{2} = \delta_\varepsilon^{(+)}$$

$\delta_\varepsilon = \delta_\varepsilon^{(+)}$ (i.e. 

)

$$\delta_\varepsilon^{(-)} = \sqrt{2}-\sqrt{2-\varepsilon} > \sqrt{2+\varepsilon}-\sqrt{2} = \delta_\varepsilon^{(+)}$$

$$2\sqrt{2} > \sqrt{2+\varepsilon}+\sqrt{2-\varepsilon}$$

$$4\cdot 2 > 2+\cancel{\varepsilon}+2-\cancel{\varepsilon}+2\sqrt{4-\varepsilon^2}$$

$$4 > 2\sqrt{4-\varepsilon^2} \qquad 2 > \sqrt{4-\varepsilon^2}$$

VERO $\forall\, \varepsilon \le \varepsilon_0 = 2$

$$4 > 4-\varepsilon^2$$

VERO $\forall\, \varepsilon > 0$

(*) $$\delta_\varepsilon = \min\left\{\delta_\varepsilon \,,\, \delta_\varepsilon^{(+)}\right\}$$

$$\forall\, \varepsilon > 0 \quad \exists\, \delta_\varepsilon > 0 : |x^2-2| < \varepsilon$$

$$\forall\, x \in \left(\sqrt{2}-\delta_\varepsilon, \sqrt{2}+\delta_\varepsilon\right) \setminus \{\sqrt{2}\}$$

VERSIONE FACILE:

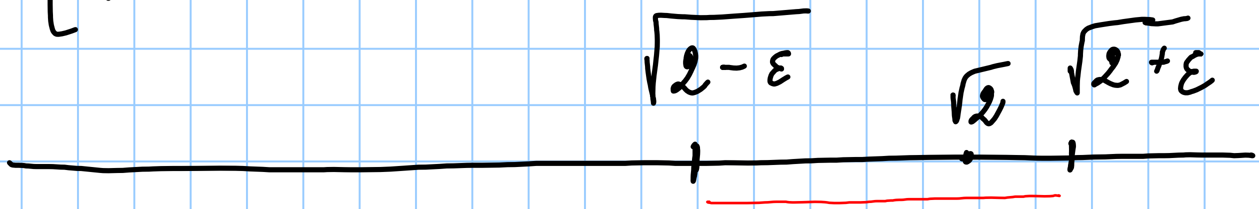
$$\forall \varepsilon > 0 \ \exists\, \mathcal{U}_{\sqrt{2},\varepsilon} : |x^2 - 2| < \varepsilon \quad \forall x \in \mathcal{U}_{\sqrt{2},\varepsilon} \setminus \{\sqrt{2}\}$$

$$|x^2-2| < \varepsilon \Longleftrightarrow 2-\varepsilon < x^2 < 2+\varepsilon$$

$$\boxed{\varepsilon \le 2}\ ; \quad \{x^2 > 2-\varepsilon\} \cap \{x^2 < 2+\varepsilon\}$$

$$\{x < -\sqrt{2-\varepsilon}\} \cup \{x > \sqrt{2-\varepsilon}\}$$

$$\{-\sqrt{2+\varepsilon} < x < \sqrt{2+\varepsilon}\}$$



$$\mathcal{U}_{\sqrt{2},\varepsilon} = \left(\sqrt{2-\varepsilon}, \sqrt{2+\varepsilon}\right)$$

$\blacksquare$

TEOREMA di Equivalenza delle definizioni di limite.

(A) $\lim_{n} f(a_n) = L$, $\forall$ $a_n \to x_0$, $n \to +\infty$
$a_n \neq x_0$

(B) $\forall V_L \ \exists U_{x_0} : f(x) \in V_L$
$\forall x \in \{U_{x_0} \cap X\} \setminus \{x_0\}$

(B) $\Longrightarrow$ (A) FACOLTATIVA

TRATTIAMO SOLO il caso $\boxed{L \in \mathbb{R}}$

PER ASSURDO, (A) E' FALSO

$\exists \{a_n\} : a_n \xrightarrow[n \to +\infty]{} x_0$, $a_n \neq x_0$

e $f(a_n)$ NON TENDE a L

$\cancel{\varnothing}$ $\quad \forall \varepsilon > 0 \ \exists \nu_\varepsilon : |f(a_n) - L| < \varepsilon, \ \forall n > \nu_\varepsilon$

$$\exists \varepsilon_0 : \forall \nu \in \mathbb{N}, \ \exists n_0 > \nu : |f(a_{n_0}) - L| \geq \varepsilon_0$$

$$\nu_1 = 1 \longrightarrow n_1 > \nu_1 = 1 : |f(a_{n_1}) - L| \geq \varepsilon_0$$

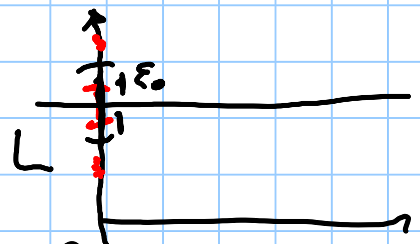
$$\nu_2 = n_1 + 1 \longrightarrow n_2 > \nu_2 = n_1 + 1 : |f(a_{n_2}) - L| \geq \varepsilon_0$$

$$\nu_3 = n_2 + 1 \longrightarrow n_3 > \nu_3 = n_2 + 1 : |f(a_{n_3}) - L| \geq \varepsilon_0$$

PER INDUZIONE

$$\exists \{a_{n_k}\} \subset \{a_n\} : |f(a_{n_k}) - L| \geq \varepsilon_0 \quad \forall k \in \mathbb{N}$$

SOTTOSUCCESSIONE



Dato che per ipotesi $a_n \longrightarrow x_0, \ n \longrightarrow +\infty$,

ALLORA $a_{n_k} \longrightarrow x_0, \ k \longrightarrow +\infty$

$V_0 = \left(L - \frac{\varepsilon_0}{4}, L + \frac{\varepsilon_0}{4}\right)$ per Ipotesi (B)

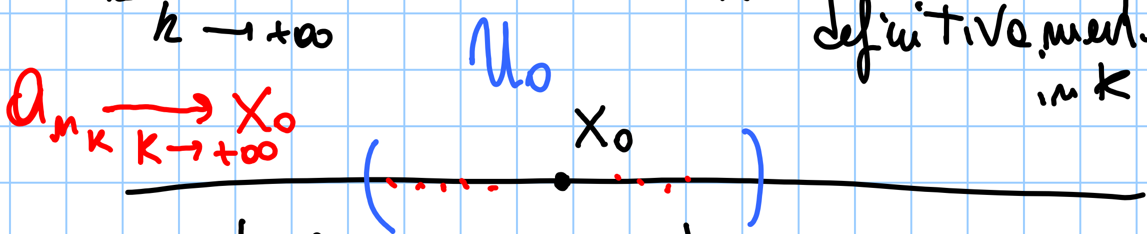
$\exists\, U_0$ di x_0 :

$f(x) \in V_0 \quad \forall x \in \{U_0 \cap X\} \setminus \{x_0\}$

OVVERO $|f(x) - L| \le \frac{\varepsilon_0}{4}$

$\forall x \in \{U_0 \cap X\} \setminus \{x_0\}$

$a_{m_k} \xrightarrow[k \to +\infty]{} x_0 \implies a_{m_k} \in U_0$ definitivamente in k



$\varepsilon_0 \le |f(a_{m_k}) - L| \underset{(B)}{\le} \frac{\varepsilon_0}{4}$ ASSURDO

(A) $\implies$ (B)

PER ASSURDO, (B) E' FALSO

(B)

$\forall V_L \ \exists \mathcal{U}_{x_0} : f(x) \in V_L \quad \forall x \in \{\mathcal{U}_{x_0} \cap X\} \setminus \{x_0\}$

($\cancel{B}$)

$\exists V_0 : \forall \mathcal{U}$ di x_0 $\exists x_u \in \{\mathcal{U} \cap X\} \setminus \{x_0\}$

$$f(x_u) \notin V_0$$

$\exists \varepsilon_0 > 0 : \quad (L - \varepsilon_0, L + \varepsilon_0) \subset V_0$

$\forall \delta > 0 \quad \exists x_\delta \in \{(x_0 - \delta, x_0 + \delta) \cap X\} \setminus \{x_0\} :$

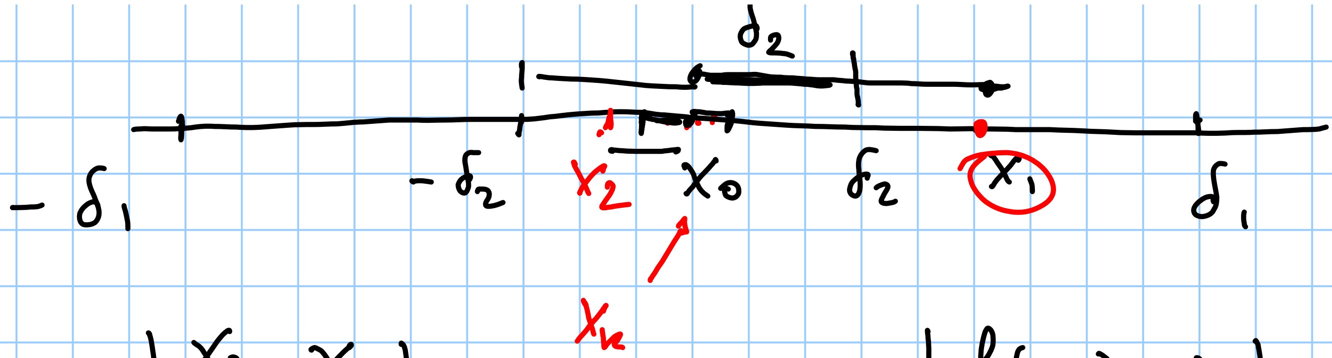
$$f(x_\delta) \notin (L - \varepsilon_0, L + \varepsilon_0)$$

$$\Updownarrow$$

$$|f(x_\delta) - L| \geq \varepsilon_0$$

$\delta_1 = 1 \longrightarrow x_1 = x_{\delta_1} : |f(x_1) - L| \geq \varepsilon_0$

$\delta_2 = \dfrac{|x_1 - x_0|}{2} \longrightarrow x_2 = x_{\delta_2} : |f(x_2) - L| \geq \varepsilon_0$



$$\delta_3 = \frac{|x_2 - x_0|}{2} \longrightarrow x_3 = x_{\delta_3} : |f(x_3) - L| \geq \varepsilon_0$$

PER INDUZIONE OTTENIAMO

$$x_k, k \in \mathbb{N} : \left| f(x_k) - L \right| \geq \varepsilon_0$$

$$|x_3 - x_0| < \delta_3 = \frac{|x_2 - x_0|}{2} < \frac{\delta_2}{2} = \frac{|x_1 - x_0|}{4} = \frac{1}{4}$$

$$|x_k - x_0| < \dots < \frac{1}{2^{k-1}} \quad \forall k \in \mathbb{N}$$

$$\lim_{k \to +\infty} x_k = x_0 \quad \left(|x_k - x_0| \xrightarrow[k \to +\infty]{} 0 \Rightarrow x_k - x_0 \xrightarrow[k \to +\infty]{} 0 \right)$$

$$\begin{cases} |f(x_k) - L| \geq \varepsilon_0 \\ \exists\, \nu_0 : |f(x_k) - L| \leq \frac{\varepsilon_0}{2} \quad \forall k > \nu_0 \end{cases}$$

ASSURDO

A

$\blacksquare$

TEOREMA

$$\begin{cases} \lim_{x\to+\infty} x^{\alpha} = +\infty, \quad \lim_{x\to+\infty} \left(\log(x^{\gamma})\right)^{\beta} = +\infty \\ \lim_{x\to+\infty} e^{\beta x^{\gamma}} = +\infty \end{cases} \quad \boxed{\alpha>0,\ \beta>0,\ \gamma>0}$$

$$\begin{cases} \lim_{x\to+\infty} \dfrac{x^{\alpha}}{\left(\log(x^{\gamma})\right)^{\beta}} = +\infty \\ \lim_{x\to+\infty} \dfrac{e^{\beta x^{\gamma}}}{x^{\alpha}} = +\infty \end{cases}$$

$$\lim_{x\to+\infty} \frac{x^{\alpha x}}{e^{\beta x^{\gamma}}} = \begin{cases} +\infty, & \gamma \leq 1 \\ 0, & \gamma > 1 \end{cases}$$

$$x^{\alpha x} = e^{\alpha x \log(x)}$$

$$\lim_{x \to +\infty} \frac{e^{\beta (\log x)^\gamma}}{x^\alpha} = \begin{cases} +\infty, & \gamma > 1 \\ \begin{cases} +\infty, & \beta > \alpha \\ 1, & \beta = \alpha \\ 0, & \beta < \alpha \end{cases} & \gamma = 1 \\ 0, & \gamma \in (0,1) \end{cases}$$

$$x^\alpha = e^{\alpha \log x}$$

VALGONO IN PARTICOLARE PER I LIMITI DI FUNZIONI

TUTTI I TEOREMI SU SOMMA/PRODOTTO e SU POTENZA/ESPONENZIALE/LOGARITMI (VEDERE LE NOTE)

$$(a+b)(a-b)=a^2-b^2$$

$$\lim_{x\to+\infty} \underbrace{\overset{a}{\sqrt{x+1}}}_{+\infty} - \underbrace{\overset{b}{\sqrt{x}}}_{-(+\infty)} =$$

$$\lim_{x\to+\infty} \frac{(x+1)-(x)}{\sqrt{x+1}+\sqrt{x}} = \frac{1}{+\infty} = 0$$

$$\lim_{x\to+\infty} \sqrt[4]{x+1} - \sqrt[4]{x} =$$

$$= \lim_{x\to+\infty} \frac{\sqrt{x+1} - \sqrt{x}}{\sqrt[4]{x+1} + \sqrt[4]{x}} = \frac{0}{+\infty} = 0$$

$$\lim_{x\to+\infty} \left(1+\frac{4}{x}\right)^x = e^4$$

APPENDICE

Una applicazione della formula della somma di potenze $\sum_{K=0}^{N} x^K = \frac{1-x^{N+1}}{1-x}$

- $\lim_{x \to 1} \frac{1-x^{N+1}}{1-x} = \lim_{x \to 1} \sum_{K=0}^{N} x^K =$

$$= \sum_{K=0}^{N} (1)^K = N+1$$

$$\lim_{x \to +\infty} \left(\sqrt[3]{x+1} - \sqrt[3]{x} \right) = 0$$

$$a^3 - b^3 = (a-b)(\;?\;)$$

$$\frac{1-x^3}{1-x} = \sum_{k=0}^{2} x^k = 1 + x + x^2$$

$$x = b/a; \quad \frac{a^3-b^3}{a-b} = a^2 + ab + b^2$$

$$\sqrt[3]{x+1} - \sqrt[3]{x} = \left\{ a = \sqrt[3]{x+1},\ b = \sqrt[3]{x} \right\}$$

$$= \frac{\left(\sqrt[3]{x+1} - \sqrt[3]{x}\right)\left((x+1)^{2/3} + (x+1)^{1/3}x^{1/3} + x^{2/3}\right)}{\left((x+1)^{2/3} + (x+1)^{1/3}x^{1/3} + x^{2/3}\right)} =$$

$$= \frac{\left(\sqrt[3]{x+1}\right)^3 - \left(\sqrt[3]{x}\right)^3}{\left((x+1)^{2/3} + (x+1)^{1/3}x^{1/3} + x^{2/3}\right)} =$$

$$= \frac{\cancel{x}+1-\cancel{x}}{(x+1)^{2/3}+(x+1)^{\frac{1}{3}}x^{\frac{1}{3}}+x^{2/3}} \xrightarrow{x\to+\infty} \frac{1}{+\infty} = 0$$

$$\lim_{x\to+\infty} \sqrt[N]{x+1} - \sqrt[N]{x} =$$

$$\lim_{x\to+\infty} \frac{1}{\sum_{k=0}^{N-1} (x+1)^{k/N} x^{\frac{N-k-1}{N}}} = \frac{1}{+\infty} = 0$$