ABBIAMO VISTO CHE $a^{b_n} \xrightarrow[n \to +\infty]{} a^b$ Se $a \in (0,+\infty) \setminus \{1\}$, $b_n \to b \in \mathbb{R}$

DIMOSTRIAMO ORA CHE

Se $b_n \longrightarrow -\infty$, $a > 1$ si ha

$$a^{b_n} \longrightarrow 0, \quad n \longrightarrow +\infty$$

$$a^{b_n} = e^{b_n \log(a)}, \quad c_n = b_n \log(a)$$

$c_n \longrightarrow -\infty$ (PERCHE' $a > 1 \Longrightarrow \log(a) > 0$)

S.P.D.G. $a = e$

Quindi E' SUFFICENTE

DIM. CHE

$$\lim_{n} e^{b_n} = 0 \text{ se } b_n \to -\infty.$$

$\forall \varepsilon > 0 \ \exists \nu_\varepsilon : -\varepsilon < e^{b_n} < \varepsilon \ \forall n > \nu_\varepsilon$

$-\varepsilon < e^{b_n}$ E' VERA $\forall n \in \mathbb{N}$

$e^{b_n} < \varepsilon \iff b_n < \log(\varepsilon)$

E' SUFFICENTE DIM. CHE

$\forall \varepsilon > 0 \ \exists \nu_\varepsilon : \ b_n < \log(\varepsilon), \ \forall n > \nu_\varepsilon$

dato che $b_n \to -\infty$ si ha

$\forall R>0 \; \exists \nu_R \in \mathbb{N}: \; b_n < -R \quad \forall n > \nu_R$

S.P.D.G. $\varepsilon \in \left(0, \frac{1}{2}\right]$ e poniamo

$$R_\varepsilon = -\log(\varepsilon).$$

Quindi $R_\varepsilon > 0$ e poniamo

$$\nu_\varepsilon = \nu_{R_\varepsilon}$$

Se $n > \nu_\varepsilon = \nu_{R_\varepsilon}$ si ha

$$b_n < -R_\varepsilon = \log(\varepsilon).$$

Quindi

$\forall \varepsilon > 0 \; \exists \nu_\varepsilon: \; b_n < \log(\varepsilon) \quad \forall n > \nu_\varepsilon$

$\blacksquare$

DIMOSTRIAMO CHE

$$\lim_{n} \log(a_n) = -\infty \quad \text{se}$$

$$a_n \to 0, \quad a_n > 0 \text{ definitivamente}$$

$$\forall M > 0 \ \exists \nu_M : \quad \log(a_n) < -M \quad \forall n > \nu_M$$

$$\log(a_n) < -M \iff 0 < a_n < e^{-M}$$

$$\forall \sigma > 0 \ \exists \mu_\sigma \in \mathbb{N} : 0 < a_n < \sigma, \ \forall n > \mu_\sigma$$

$$\Updownarrow$$

$$a_n \to 0 \text{ e } a_n > 0 \text{ definitivamente}$$

$$\nu_M = \mu_{e^{-M}} = \mu_\sigma\big|_{\sigma=e^{-M}}$$

$$\forall M>0 \ \exists \nu_M : 0 < a_n < e^{-M} \quad \forall n > \nu_M$$

$\blacksquare$

DIAMO PER ACQUISITO CHE

$$a^{b_n} \longrightarrow +\infty \quad \text{se} \quad b_n \longrightarrow +\infty .$$

$$\log(a_n) \longrightarrow \log(a) \quad \text{se } a_n \longrightarrow a$$
$$a \in (0,+\infty)$$

ALLORA SI HA:

$$a_n^{b_n} \longrightarrow a^b \quad , \quad a_n \to a \in (0,+\infty)\setminus\{1\}$$
$$b_n \to b \in [-\infty,+\infty]$$

Infatti, osserviamo che

$$a_n \to a \in (0,+\infty) \setminus \{1\}$$

$$a_n^{b_n} = e^{b_n \log(a_n)}, \quad \log(a_n) \longrightarrow \log(a)$$

$$\log(a) \in \mathbb{R} \setminus \{0\}$$

$$b_n \log(a_n) \longrightarrow b \log(a) \in [-\infty, +\infty]$$

$$a_n^{b_n} = e^{b_n \log(a_n)} \longrightarrow e^{b \log(a)}$$

$$= \begin{cases} 0 & \text{se } b\log(a) = -\infty \\ a^b & \text{se } b\log(a) \in \mathbb{R} \\ +\infty & \text{se } b\log(a) = +\infty \end{cases}$$

$$\blacksquare$$

$$\lim_{n} e^{\frac{1}{n}} = e^{0} = 1$$

$$\lim_{n} \Big(\underset{\substack{\vee \\ 0}}{\frac{1}{n}} \Big)^{n} = (0^{+})^{+\infty} = 0$$

$$\lim_{n} \left(\underbrace{\frac{5n^2+1}{n^2+e^{-n}}}_{\to 5} \right)^{\pi} = 5^{\pi}$$

$a_n^{\,b_n}$ $\quad b_n = \pi$

$a_n \longrightarrow 5$

$$\lim_{n} \left(\frac{1}{n^{100}}\right)^{\frac{1}{n}} \overset{?}{=} 0^0$$

$$\left(\frac{1}{n^{100}}\right)^{\frac{1}{n}} = e^{\overset{100}{\frac{1}{n}}\log\left(\frac{1}{n}\right)} = e^{-\overset{100}{}\frac{\log(n)}{n}}$$

$$= e^{-\overset{100}{}\frac{\log(n)}{n}} \longrightarrow e^0 = 1$$

$$\lim_{n} \left(n^{100}\right)^{1/n} \overset{?}{=} +\infty^0$$

$$n^{\frac{1}{n}} = e^{\frac{1}{n}\log(n)} = e^{\frac{\log(n)}{n}} \longrightarrow e^0 = 1$$

$$\lim_{n} \left(e^{-n^4}\right)^{1/n} = \lim_{n} e^{-n^3} = e^{-\infty} = 0$$

$$\frac{2^{n^2}}{n^n} = \frac{2^{n^2}}{2^{n\log_2(n)}} = 2^{n^2 - n\log_2(n)}$$

$$= 2^{n^2\left(1-\frac{\log_2(n)}{n}\right)} \longrightarrow 2^{+\infty(1-0)}$$

$$= 2^{+\infty} = +\infty$$

$$\frac{2^{\sqrt{n}}}{n^n} = 2^{\sqrt{n} - \underline{n\log_2(n)}} =$$

$$\begin{cases} \sqrt{n} - \underbrace{n \log_2(n)} = n\log_2(n)\left(-1+o(1)\right) \\ \dfrac{\sqrt{n}}{n\log_2(n)} = \dfrac{1}{\sqrt{n}\cdot\log_2(n)} \to 0 \end{cases}$$

$$= 2^{-n\log_2(n)(1+o(1))} \longrightarrow 2^{-\infty} = 0$$

$$\left(\frac{2}{n}\right)^n = \frac{2^n}{n^n} \longrightarrow (0^+)^{+\infty} = 0$$

$$a > 1$$

$$\frac{a^{n^\gamma}}{n^n} \longrightarrow \begin{cases} +\infty & \gamma > 1 \\ 0 & \gamma \in (0,1] \end{cases}$$

$$\lim \left(1+\frac{1}{n^2}\right)^n \overset{?}{=} 1^{+\infty} ?$$

$$\lim_n \left(1+\frac{1}{a_n}\right)^{a_n} = e$$

$a_n \longrightarrow +\infty$

$a_n \longrightarrow -\infty$

$$\left(1+\frac{1}{n^2}\right)^n =$$

$$= \left(1+1/n^2\right)^{n^2/n} =$$

$$\Bigg[\underbrace{\left(1+\frac{1}{n^2}\right)^{n^2}}_{\to e}\Bigg]^{\frac{1}{n} \to 0} \longrightarrow e^0 = 1.$$

$$\lim_n \left(1+\frac{1}{n^2}\right)^{n^4} =$$

$$= \lim_n \Bigg[\underbrace{\left(1+\frac{1}{n^2}\right)^{n^2}}_{\to e}\Bigg]^{n^2 \to +\infty} = e^{+\infty} = +\infty$$

$$\lim_{n}\left(1+\frac{5}{n^2}\right)^{n^2}=\lim_{n}\left(1+\frac{1}{\frac{n^2}{5}}\right)^{\frac{n^2}{5}\cdot 5}=$$

$$=\lim_{n}\left[\left(1+\frac{1}{n^2/5}\right)^{n^2/5}\right]^5=e^5$$

$$\lim_{n}\left(1+\frac{x}{n}\right)^{n}=e^{x}\quad,\ x\in\mathbb{R}$$

$$\lim_{n}\left(\frac{n^2+n+1}{n^2+5}\right)^{3n}=\lim_{n}\left(1+\frac{n-4}{n^2+5}\right)^{3n}$$

$$c_n\to 1\Longrightarrow c_n=1+o(1)$$

$$1+\frac{n^2+n+1}{n^2+5}-1=1+\frac{n^2+n+1-n^2-5}{n^2+5}=$$

$$=1+\frac{n-4}{n^2+5}\ ;\ a_n=\frac{n^2+5}{n-4}$$

$$\left(1+\frac{n-4}{n^2+5}\right)^{3n}=\left(1+\frac{1}{a_n}\right)^{a_n\cdot\frac{n-4}{n^2+5}\cdot 3n}=$$

$$= \Bigg[\underbrace{\left(1 + \frac{1}{a_n}\right)^{a_n}}_{\to e} \Bigg]^{\overbrace{\frac{(n-4)\,3n}{n^2+5}}^{\to 3}} \longrightarrow e^3$$

$$a_n = \frac{n^2+5}{n-4} \longrightarrow +\infty$$

$$\lim_n \left(\frac{4}{n} + 2^{-n} \right)^{\frac{1}{n}} \overset{?}{=} 0^0 \ ?$$

$$\left(\frac{4}{n} + 2^{-n} \right)^{1/n} = e^{\frac{1}{n} \log\left(\frac{4}{n} + 2^{-n}\right)} =$$

$$\log\left(\frac{4}{n} + 2^{-n} \right) =$$

$$= \log\left(\frac{4}{n} \left(1 + \frac{n 2^{-n}}{4} \right) \right) =$$

$$= \log \frac{4}{n} + \log\left(1 + \frac{n 2^{-n}}{4} \right) =$$

$$= \left(\log \frac{4}{n} \right) \left(1 + o(1) \right) \doteq$$

NOTA

$$o(1) = \frac{\log\left(1+\frac{m}{4}2^{-m}\right)}{\log 4/m} =$$

$$= \frac{\log(1+o(1))}{\log 4 - \log(m)} \longrightarrow \frac{\log(1)}{-\infty} =$$

$$= \frac{0}{-\infty} = 0$$

$$e^{\frac{1}{m}\log\left(\frac{4}{m}+2^{-m}\right)} = e^{\frac{1}{m}\left(\log\frac{4}{m}\right)\left(1+o(1)\right)}$$

$$= e^{-\frac{\log\frac{m}{4}}{m}\left(1+o(1)\right)} \longrightarrow e^{0\cdot 1} = e^{0} = 1$$

$$\log\left(\frac{4}{m}\right) = \log\left(\frac{m}{4}\right)^{-1}$$

$$\lim_{n} \left(\frac{1}{n} + 3^{-n} \right)^{n} = 0^{+\infty} = 0$$

$$\lim_{n} \left(\frac{1}{n} + 3^{-n} \right)^{\frac{1}{n}} \overset{?}{=} 0^{0} \;?$$

$$\left(\frac{1}{n} + 3^{-n} \right)^{\frac{1}{n}} = e^{\frac{1}{n} \log\left(\frac{1}{n} + 3^{-n} \right)}$$

$$\frac{1}{n} + 3^{-n}$$

$a_n = \frac{1}{n} = o(1)$, $b_n = 3^{-n} = o(1)$

COME INDIVIDUO IL TERMINE dominante?

Calcoliamo il limite del rapporto $\lim_n \frac{b_n}{a_n} =$

$$= \lim_n \frac{3^{-n}}{1/n} = \lim_n \frac{n}{3^n} = 0$$

b_n è molto più piccolo di a_n per $n \to +\infty$.

Se $a_n = o(1)$, $b_n = o(1)$ e $\lim_n \frac{b_n}{a_n} = 0$

Si dice che b_n è un

INFINITESIMO di ORDINE SUPERIORE

ad a_n, $b_n = o(a_n)$

$$\frac{1}{n} + 3^{-n} = \frac{1}{n}\left(1 + o(1)\right)$$

$$\log\left(\frac{1}{n} + 3^{-n}\right) = \log\left(\frac{1}{n}\left(1+o(1)\right)\right)$$

$$= \log\left(\frac{1}{n}\right) + \log\left(1+o(1)\right) =$$

$$= \log\left(\frac{1}{n}\right)\left(1 + \underbrace{\frac{\log(1+o(1))}{-\log(n)}}_{=o(1)}\right)$$

$$e^{\frac{1}{n}\log\left(\frac{1}{n}+3^{-n}\right)} = e^{\frac{1}{n}\left(-\log(n)+o(1)\right)}$$

$$= e^{-\frac{\log(n)}{n}\left(1+o(1)\right)} \to e^{0\cdot 1} = 1$$

ANALOGAMENTE se

$a_n \to +\infty$, $b_n \to +\infty$ e se

$\lim_n \frac{b_n}{a_n} = +\infty$ si dice b_n

è un **INFINITO di ORDINE SUPERIORE** ad a_n,

e b_n è molto più grande di

a_n per $n \to +\infty$.

In particolare $\lim_n \frac{a_n}{b_n} = 0$ e

si scrive $a_n = o(b_n)$

PER EVITARE di FARE CONFUSIONE RICORDARE quanto SEGUE:

Se $a_n = o(1)$ e $b_n = o(1)$

o $a_n \to \pm\infty$ e $b_n \to \pm\infty$

e se $\lim_n \frac{a_n}{b_n} = 0$,

allora b_n è molto più grande di a_n per $n \to +\infty$.

$$\lim_n \frac{\sqrt{n}\log(n) + n}{e^n + n^2\log(n)} =$$

$$\begin{cases} \lim\limits_{n} \dfrac{\sqrt{n}\log(n)}{n} = \lim\limits_{n} \dfrac{\log(n)}{\sqrt{n}} = 0 \\ \lim\limits_{n} \dfrac{n^2\log(n)}{e^n} = \lim\limits_{n} \dfrac{n^3}{e^n}\dfrac{\log(n)}{n} = 0 \end{cases}$$

$$= \lim_{n} \frac{n(1+o(1))}{e^n(1+o(1))} = 0$$

APPENDICE

$$\frac{e^{a_n}}{n} \qquad (\log(n))^\alpha$$

$\alpha > 0$

$$\alpha = 1; \quad \frac{e^{\log n}}{n} = \frac{n}{n} = 1.$$

$$\frac{e^{(\log(n))^\alpha}}{n} = e^{(\log(n))^\alpha - \log(n)} \begin{cases} \to +\infty, & \alpha > 1 \\ \to 1, & \alpha = 1 \\ \to 0, & \alpha < 1 \end{cases}$$

FORMULA DI STIRLING

$$n! = n^n e^{-n} \sqrt{2\pi n}\,\left(1 + o(1)\right)$$

$$\lim_{n} \frac{n!}{n^n e^{-n} \sqrt{2\pi n}} = 1$$

ATTENZIONE

$$n! = n^n e^{-n} \sqrt{2\pi n}\left(1 + \frac{1}{12n} + o\left(\frac{1}{n}\right)\right)$$

$$= n^n e^{-n} \sqrt{2\pi n} + \underbrace{\frac{n^n e^{-n} \sqrt{2\pi n}}{12n}\left(1+o(1)\right)}$$

L'ERRORE per $n \to +\infty$ è UN NUMERO GRANDISSIMO

$$\lim_n \frac{n^{\alpha n}}{n!} = \lim_n \frac{n^{(\alpha-1)n}\, e^n}{\sqrt{2\pi n}\left(1+o(1)\right)} =$$

$$= \begin{cases} \alpha \geq 1 & +\infty \\ \alpha < 1 & 0 \end{cases}$$

$\alpha - 1 < 0$ $\alpha - 1 < 0$

$$\frac{m^{(\alpha-1)m} e^{m}}{\sqrt{2\pi m}} = \frac{e^{(\alpha-1)(\log m)m + m}}{\sqrt{2\pi m}} =$$

$$= \frac{e^{(\alpha-1)m\log(m)(1+o(1))}}{\sqrt{2\pi m}} \longrightarrow \frac{e^{-\infty}}{+\infty} = 0$$

$$\lim_{n} \frac{(2n)!}{n^n} = \lim_{n} \frac{(2n)^{2n} e^{-2n} \sqrt{4\pi n}\,(1+o(1))}{n^n}$$

$$= \lim_{n} 4^n \frac{n^n}{e^{2n}} \sqrt{4\pi n} = +\infty .$$

CRITERIO RAPPORTO PER LIMITI DI SUCCESSIONI

Se $a_n > 0$ definitivamente e

se $\lim_{n} \frac{a_{n+1}}{a_n} = a$, allora

Se $a \in [0,1)$ si ha $\lim_{n} a_n = 0$

Se $a > 1$ si ha $\lim_{n} a_n = +\infty$

$$\lim_{n} \frac{n!}{n^n}$$

$$\lim_{n} \frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}} = \lim_{n} \frac{(n+1)!}{n!} \frac{n^n}{(n+1)^{n+1}} =$$

$$= \lim_{n} \cancel{(n+1)} \frac{1}{\left(1+\frac{1}{n}\right)^n} \frac{1}{\cancel{(n+1)}} = \frac{1}{e} < 1$$

$$a = \frac{1}{e} < 1 \implies \lim_{n} \frac{n!}{n^n} = 0$$