TEOREMA

Siano a_n e b_n due SUCCESSIONI

Supponiamo CHE:

$$a_n \longrightarrow +\infty \ (-\infty), \quad n \longrightarrow +\infty$$

e ANCHE CHE $\exists \nu$:

$$b_n \geq a_n \ (b_n \leq a_n) \quad \forall n > \nu.$$

ALLORA $b_n \longrightarrow +\infty \ (-\infty)$,

$n \longrightarrow +\infty$

DIM.

$$b_n \geq a_n \quad \forall n > \nu.$$

$$\forall M>0 \ \exists\, \nu_M : a_n > M \quad \forall n > \nu_M$$

PER $M>0$ FISSATO,

SE $n > \nu$ e SE $n > \nu_M$

$$b_n \geq a_n > M$$

$$\nu_M^{(b)} := \max\{\nu, \nu_M\}$$

$\forall M > 0$ SI HA

$$b_n > M \quad \forall n > \nu_M^{(b)}$$

$$b_n = \underbrace{-3^n}_{\to\, -\infty} + \underset{\to\, ???}{\sin\left(n^2 + \log_2(n)\right)}$$

$$b_n = -3^n + \sin\left(n^2 + \log_2(n)\right)$$

$$\leq -3^n + 1 \,, \quad \forall n \in \mathbb{N}$$

$$a_n = -3^n + 1 \longrightarrow -\infty + 1 = -\infty$$

$$\Longrightarrow \quad b_n \longrightarrow -\infty \,, \quad n \longrightarrow +\infty$$

$$a^{\beta n^\gamma} \longrightarrow +\infty \,, \quad n \longrightarrow +\infty \,. \qquad a > 1, \beta > 0, \gamma > 0$$

$$\forall M > 0 \ \exists \nu_M : \ a^{\beta n^\gamma} > M \,, \quad \forall n > \nu_M$$

$$a^{\beta m^{\gamma}} > M \quad (\log_a x \nearrow)$$

$$\beta m^{\gamma} > \log_a(M)$$

$$\nu_M = 1 \quad \text{se } M \in (0,1]$$

Se $M > 1$, $\quad m^{\gamma} > \frac{1}{\beta} \log_a M$

$$= \log_a\left(\sqrt[\beta]{M}\right)$$

$(x^{1/\gamma} \nearrow) \quad m > \left(\log_a\left(\sqrt[\beta]{M}\right)\right)^{1/\gamma}$

$$\nu_M = \left[\left(\log_a\left(\sqrt[\beta]{M}\right)\right)^{1/\gamma}\right] + 1$$

$$M > 1$$

$$\frac{2^{\sqrt{n}}}{n} = 2^{\sqrt{n} - \log_2(n)} =$$

$$= 2^{\sqrt{n}\left(1 - \frac{\log_2(n)}{\sqrt{n}}\right)}$$

$$= 2^{\sqrt{n}\left(1 + o(1)\right)} \geq$$

$\Big($ $o(1)$ = qualunque successione infinitesima

$o(1) \geq -\frac{1}{2} \quad \forall n > \nu_1$ $\Big)$

$$\geq 2^{\sqrt{n}/2} \quad \forall n > \nu_1$$

CON I TEOREMI DI CONFRONTO OTTENIAMO STIME MIGLIORI

$o(1) \geq -\frac{1}{4}$ $\forall n > \nu$ $\Longrightarrow$ $2^{\sqrt{n}(1+o(1))} \geq 2^{\frac{3}{4}\sqrt{n}} \quad \forall n > \nu.$

$a_n = o(b_n)$ se $a_n \longrightarrow +\infty$, $b_n \longrightarrow +\infty$ e

$$\lim_n \frac{a_n}{b_n} = 0.$$

$$\lim_n \frac{\log_a(n)}{\sqrt[4]{n}} = 0$$

$$\forall \varepsilon > 0 \ \exists \nu_\varepsilon : 0 \leq \frac{\log_a(n)}{\sqrt[4]{n}} < \varepsilon$$

Sia $\varepsilon = 1$ $\quad \log_a(n) < \sqrt[4]{n}$, $n > \nu_1$

$$\frac{2^{\sqrt{n}}}{2^n} = 2^{\sqrt{n} - \log_a(n)} > 2^{\sqrt{n} - \sqrt[4]{n}} = 2^{\sqrt{n}\left(1 - \frac{1}{\sqrt[4]{n}}\right)} \quad \forall n > \nu_1$$

CON I TEOREMI DI CONFRONTO POSSIAMO OTTENERE ALTRI LIMITI (NON COMPRESI NEL TEOREMA DI LEZIONE 11)

$$b_n = \frac{2^{n^2}}{n^n} = 2^{n^2 - n\log_2 n} =$$

$$= 2^{n^2\left(1 - \frac{\log_2(n)}{n}\right)} =$$

$$= 2^{n^2(1+o(1))} \geq 2^{\frac{n^2}{2}} \quad \forall n > N$$

$$\boxed{1+o(1) \geq \frac{1}{2} \quad \forall n > N}$$

$$= a_n \longrightarrow +\infty, \quad n \to +\infty$$

$$b_n \geq a_n \Longrightarrow b_n \longrightarrow +\infty$$

$$\boxed{b_n = \frac{2^{\sqrt{n}}}{n^n} \longrightarrow \ ? \qquad \text{Ex}}$$

TEOREMA DI CONFRONTO

TEOREMA [PER LIMITI DI SUCC.]

SIANO a_n, b_n, c_n Tre successioni:

$$a_n \to L_1 \in \overline{\mathbb{R}}$$
$$b_n \to L_2 \in \overline{\mathbb{R}}$$

e

$$a_n \leq c_n \leq b_n \,, \; n > \nu .$$

(i) SE $c_n \to c \in \overline{\mathbb{R}}$

(c_n è REGOLARE)

ALLORA

$$L_1 \leq c \leq L_2$$

(ii) Se $L_1 = L_2 \ (=: L)$.

ALLORA c_n è REGOLARE

e $c_n \longrightarrow L$, $n \longrightarrow +\infty$

DIM.

(i) DIMOSTRIAMO CHE: $\boxed{L_1 \leq c}$

Se $L_1 = -\infty \Longrightarrow L_1 \leq c$

S.P.D.G. $L_1 > -\infty$

SE $L_1 = +\infty \Longrightarrow$ (TEOREMA PRECEDENTE)

$c = +\infty$ e $L_1 = c$.

SE $c = +\infty$ $\quad L_1 \leq c$

S.P.D.G. $L_1 \in \mathbb{R}$ e $c < +\infty$

$d_n = a_n - c_n$

PER IPOTESI $d_n = a_n - c_n \leq 0$, $n > \nu$.

$c_n \to c$, $a_n \to L_1$

$\Rightarrow d_n \to d = L_1 - c$

SE $c = -\infty \Rightarrow d_n \to +\infty$

IMPOSSIBILE ($d_n \leq 0$ $n > \nu$)

S.P.D.G. $c \in \mathbb{R}$

$d_n \leq 0$, $d_n \to L_1 - c$

(COROLLARIO TEOREMA

PERMANENZA DEL SEGNO)

$$\Longrightarrow \quad L_1 - c \leq 0$$

$$\left(d_n \leq 0 \quad n > \nu \quad e \quad d_n \to d \right.$$

$$\left. \Longrightarrow d \leq 0 \right)$$

(ii) $\quad a_n \leq c_n \leq b_n \quad , \; n > \nu$

$$a_n \longrightarrow L \; , \; b_n \longrightarrow L$$

$$L \in \overline{\mathbb{R}}$$

SE $\quad L = +\infty \quad , \quad a_n \leq c_n$

$(a_n \longrightarrow +\infty)$

$$\Longrightarrow \; c_n \longrightarrow +\infty$$

S.P.D.G. $\quad L \in [-\infty, +\infty)$

$L = -\infty \quad , \quad c_n \leq b_n \quad , \; n > \nu$

$(b_n \to -\infty) \Longrightarrow c_n \to -\infty$

S.P.D.G. $L \in \mathbb{R}$. Sia $\boxed{\varepsilon > 0.}$

(1) $a_n \to L \Longrightarrow a_n - L > -\varepsilon, \forall n > \nu_\varepsilon^{(1)}$

(2) $b_n \to L \Longrightarrow b_n - L < \varepsilon, \forall n > \nu_\varepsilon^{(2)}$

(3) PER IPOTESI $a_n \leq c_n \leq b_n$

$n > \nu$

$\forall \varepsilon > 0 \quad \forall n > \max\{\nu_\varepsilon^{(1)}, \nu_\varepsilon^{(2)}, \nu\}$.

SI HA

$$-\varepsilon + L < a_n \leq c_n \leq b_n < L + \varepsilon$$

$\Longrightarrow \forall \varepsilon > 0$ si ha $|c_n - L| < \varepsilon$

$$\forall n > \nu_\varepsilon$$

$$\nu_\varepsilon = \max\{\nu_\varepsilon^{(1)}, \nu_\varepsilon^{(2)}, \nu\}$$

∎

$$c_n = \frac{1}{\sqrt{n}} \operatorname{arctg}\left(\frac{2^n + n}{n! + 5}\right)$$

$$\underbrace{-\frac{\pi}{2\sqrt{n}}}_{a_n \to 0} \le \frac{\operatorname{arctg}\left(\frac{2^n+n}{n!+5}\right)}{\sqrt{n}} \le \underbrace{\frac{\pi}{2\sqrt{n}}}_{b_n \to 0}$$

$$\Longrightarrow \quad c_n \to 0$$

$a_n := 0 < c_n \le b_n$, $b_n \to 0$

$a_n \to 0 \quad \Longrightarrow \quad c_n \to 0$

$$a_n = \frac{1}{n^2}, \quad c_n = \frac{1}{n^2} + 1$$

$$\begin{array}{ccc} a_n & < & c_n \ \forall n \\ \downarrow & & \downarrow \\ 0 & < & 1 \end{array}$$

$$a_n = \frac{1}{n^2}, \quad c_n = \frac{1}{n}$$

$$\begin{array}{ccc} a_n & < & c_n, \ n \geq 2 \\ \downarrow & & \downarrow \\ 0 & = & 0 \end{array}$$

COROLLARIO

Siano a_n e b_n SUCCESSIONI REGOLARI

Se $a_n \leq b_n$ o se $a_n < b_n$

$$\Longrightarrow \lim_n a_n \leq \lim_n b_n$$

$$\lim_{n} \frac{n^5 + \sin(n^5)}{4n^5 + (\cos(2^n))^3} =$$

$$\lim_{n} \frac{n^5}{n^5} \cdot \frac{1 + \underbrace{\frac{\sin(n^5)}{n^5}}_{\to 0}}{4 + \underbrace{(\cos(2^n))^3 / n^5}_{\to 0}}$$

$$\underbrace{-\frac{1}{n^5}}_{a_n \to 0} \le \frac{(\cos(2^n))^3}{n^5} \le \underbrace{\left(\frac{1}{n^5}\right)}_{b_n \to 0}$$

$$\downarrow$$
$$0$$

$$\lim_{n} \frac{1+o(1)}{4+o(1)} = \frac{1}{4}$$

a_n, $|a_n| \longrightarrow 0 \Longrightarrow a_n \longrightarrow 0$

$$\underbrace{-|a_n|}_{0} \le \underbrace{a_n}_{\downarrow\, 0} \le \underbrace{|a_n|}_{\to 0}$$

$$\boxed{a \in (-1,1), \quad a^n \xrightarrow[n \to +\infty]{} 0}$$

SE $a \in (0,1) \Longrightarrow b = 1/a > 1$

$$\Longrightarrow b^n \xrightarrow[n \to +\infty]{} +\infty \qquad a^n = \frac{1}{b^n} \longrightarrow 0$$

$$\left(\frac{1}{2}\right)^n = \frac{1}{2^n}$$

SE $a \in (-1,0) \Longrightarrow |a| \in (0,1)$

$$|a^n| = \underset{\substack{\downarrow \\ 0}}{|a|^n} \xrightarrow[n \to +\infty]{} 0 \Longrightarrow a^n \to 0$$

$$\boxed{\left(-\frac{1}{2}\right)^n} \qquad -\left(\frac{1}{2}\right)^n \leq \left(-\frac{1}{2}\right)^n \leq \left(\frac{1}{2}\right)^n$$

$$\lim_{n} \frac{(\log_a(n))^2 3^n - 2^n \sin(n)}{\left(2\log_a(n) - \cos(n)\sqrt{\log_a(n)}\right) 2^n (\ldots\ldots)} \qquad \boxed{a > 1}$$

$$\left(\left(\frac{3}{2}\right)^n - n^4 \log_a(n) \operatorname{arctg}(n)\right)$$

$$\left(\log_a(n)\right)^2 3^n - 2^n \cdot \sin(n) =$$

$$3^n \left(\log_a(n)\right)^2 \left(1 - \underbrace{\frac{2^n \sin(n)}{3^n \left(\log_a(n)\right)^2}}\right) =$$

$$3^n \left(\log_a(n)\right)^2 \left(1 + o(1)\right)$$

$$\underbrace{\frac{-1}{\left(\log_a(n)\right)^2}}_{-\frac{1}{+\infty}\cdot 0}\left(\frac{2}{3}\right)^n \le \underbrace{\left(\frac{2}{3}\right)^n \frac{\sin(n)}{\left(\log_a(n)\right)^2}}_{0} \le \left(\frac{2}{3}\right)^n \cdot \frac{1}{\left(\log_a(n)\right)^2}$$

$$0 \cdot \frac{1}{+\infty} = 0$$

$$2\log_a(n) - \cos(n)\sqrt{\log_a(n)} =$$

$$\log_a(n)\left(2 - \frac{\cos(n)}{\sqrt{\log_a(n)}}\right) =$$

$$= \log_a(n)\left(2 + o(1)\right)$$

$$\left(\frac{3}{2}\right)^n - n^4 \log_a(n) \operatorname{arcTg}(n)$$

$$\left(\frac{3}{2}\right)^n \left(1 - \frac{n^4 \log_a(n) \operatorname{arcTg}(n)}{\left(\frac{3}{2}\right)^n}\right)$$

$$0 \le \frac{n^4 \log_a(n)}{\left(\frac{3}{2}\right)^n} \le$$

$\log_a(n) = o(n)$

$\Downarrow$

$\log_a(n) \le n, \; n \gg 1$

(definitivamente) $$\frac{n^4 \cdot n}{\left(\frac{3}{2}\right)^n} = \frac{n^5}{\left(\frac{3}{2}\right)^n} \longrightarrow 0$$

$\frac{n^\alpha}{a^n} \longrightarrow 0, \quad \alpha > 0, \; a > 1$.

$$\begin{array}{c} 0 \\ \shortparallel \\ a_m \end{array} \le \underbrace{\frac{m^4 \log_a(m)\,\mathrm{arcTg}(m)}{\left(\frac{3}{2}\right)^m}}_{\downarrow \atop 0} \le \frac{m^5 \frac{\pi}{2}}{\left(\frac{3}{2}\right)^m} \to 0$$

$$\lim_m \frac{\cdots}{\cdots} =$$

$$\lim_m \frac{3^m \left(\log_a(m)\right)^2 \left(1+o(1)\right)}{\log_a(m)\left(2+o(1)\right)\cdot 2^m \cdot \left(\frac{3}{2}\right)^m \left(1+o(1)\right)} =$$

$$\lim_m \log_a(m) \cdot \frac{1}{2} = +\infty$$