$$-\infty \qquad +\infty$$

$$\lim_{n} \frac{-5n^3+n+4}{6n^7-5n^2-1}$$

$$+\infty - (+\infty)$$

$$\frac{-5n^3+n+4}{6n^7-5n^2-1} = \frac{\cancel{n^3}}{n^{\cancel{7}\,4}} \cdot \frac{\left(-5+\underbrace{\frac{n}{n^3}}_{0}+\underbrace{\frac{4}{n^3}}_{0}\right)}{\left(6-\underbrace{\frac{5n^2}{n^7}}_{0}-\underbrace{\frac{1}{n^7}}_{0}\right)}$$

$$\longrightarrow -\frac{1}{+\infty}\cdot\frac{5+0+0}{6+0+0} = 0\cdot\frac{5}{6} = 0$$

NOTAZIONE IMPORTANTISSIMA

$o(1)$ (o piccolo di 1) = qualunque successione INFINITESIMA

$$n^2-n = n^2\left(1-\frac{1}{n}\right) = n^2\left(1+o(1)\right) \longrightarrow$$

$$\longrightarrow +\infty(1+0) = +\infty, \quad n\to+\infty$$

$$n^2 - n = n(n-1) \longrightarrow +\infty$$

$$+\infty - (+\infty) \qquad \qquad +\infty \cdot (+\infty)$$

$$a_n \longrightarrow +\infty, \quad b_n \longrightarrow +\infty \Longrightarrow a_n \cdot b_n \longrightarrow +\infty$$

$$n \longrightarrow +\infty$$

SOLUZIONE FORME INDETERMINATE

$$\lim_n \left(4^n - n^5\right) = +\infty - (+\infty)$$

SE VALE: $\frac{n^5}{4^n} \longrightarrow 0, \quad n \longrightarrow +\infty$

ALLORA $4^n - n^5 = 4^n \left(1 - \frac{n^5}{4^n}\right) =$

$$= 4^n \left(1 + o(1)\right) \longrightarrow +\infty \cdot (1 + 0) = +\infty.$$

TEOREMA $\alpha > 0,\ \beta > 0,\ \gamma > 0$
$a > 1$

$$\lim_n \frac{n^\alpha}{(\log_a(n^\gamma))^\beta} = +\infty$$

$$\lim_n \frac{a^{\beta n^\gamma}}{n^\alpha} = +\infty$$

$$\lim_n \frac{(n!)^\alpha}{a^{\beta n}} = +\infty$$

$$\lim_n \frac{n^{\alpha n}}{a^{\beta n}} = +\infty$$

$$\lim_n \frac{n^n}{n!} = +\infty$$

N.B. SE DIMOSTRIAMO $\lim_n \frac{n^n}{n!} = +\infty$

ALLORA SI HA ANCHE

$$\lim_n \frac{n!}{n^n} = \lim_n \left(\frac{n^n}{n!}\right)^{-1} = \frac{1}{+\infty} = 0$$

$$\lim_{n} \frac{a^{\beta n^{\gamma}}}{n^{\alpha}} = +\infty, \quad \alpha = 1, \ \beta = 1, \ \gamma = 1$$

$$\boxed{a > 1}$$

$$\lim_{n} \frac{a^n}{n} = +\infty$$

$$\forall M > 0 \ \exists \nu_M \in \mathbb{N} : \frac{a^n}{n} > M, \ \forall n > \nu_M$$

$$a^n > \binom{n}{2}(a-1)^2 = \frac{n(n-1)}{2}(a-1)^2$$

$$n \geq 2$$

FISSATO $M > 0$,

$$\frac{a^n}{n} > \frac{(n-1)}{2}(a-1)^2 > M$$

$$n - 1 > \frac{2}{(a-1)^2} M$$

$$n > \frac{2}{(a-1)^2} M + 1$$

$$\nu_M := \nu_M^{(a)} := \left[\frac{2}{(a-1)^2} M + 1\right] + 1 \qquad (\nu_M^{(a)} \geq 2)$$

$$\boxed{\alpha > 0}$$

$$\frac{a^m}{m^\alpha} = \left(\frac{a^{m/\alpha}}{m}\right)^\alpha = \left(\frac{b^m}{m}\right)^\alpha, \quad b = a^{1/\alpha} > 1$$

$$\boxed{M>0} \quad \left(\frac{b^m}{m}\right)^\alpha > M \quad \forall\, m > \nu_M$$

$$\Updownarrow$$

$$\left(x^{\frac{1}{\alpha}} \nearrow\right) \quad \frac{b^m}{m} > M^{\frac{1}{\alpha}} . \quad m > \nu^{(b)}_{M^{\frac{1}{\alpha}}} =: \nu_M$$

$$\nu_M^{(a)} = \left[\frac{2}{(a-1)^2} M + 1\right] + 1$$

$$\nu^{(b)}_{M^{\frac{1}{\alpha}}} = \left[\frac{2}{(b-1)^2} M^{\frac{1}{\alpha}} + 1\right] + 1$$

IN PARTICOLARE: $a>1,\ \alpha\in\mathbb{R}$

$$\frac{a^n}{n^\alpha} \longrightarrow +\infty, \quad n\to+\infty$$

$$\lim_{n} \frac{n^\alpha}{\left(\log_a(n^\gamma)\right)^\beta} = +\infty$$

$\alpha=1,\ \gamma=1,\ \beta=1$

$$\lim_{n} \frac{n}{\log_a(n)} = +\infty$$

$$\forall M>0 \ \exists \nu_M : \frac{n}{\log_a(n)} > M \quad \forall n > \nu_M$$

$$\boxed{n \geq 2}$$

$$\frac{n}{\log_a(n)} > M \iff n > M \log_a(n)$$

$$\iff a^n > a^{M \log_a(n)} = a^{\log_a(n^M)} = n^M .$$

Quindi E' SUFFICENTE dim. che

$$\forall M>0 \ \exists \nu_M \quad \frac{a^n}{n^M} > 1, \ \forall n > \nu_M$$

OVVERO

$\forall M>0 \ \exists \nu_M : \frac{a^n}{n^M} > 1$ DEFINITIVAMENTE in n

$M>0$ FISSATO. $\frac{a^n}{n^M} > 1$?

$\alpha > 0, \quad \dfrac{a^n}{n^\alpha} \longrightarrow +\infty, \quad n \longrightarrow +\infty \implies$

$\forall M > 0, \quad \dfrac{a^n}{n^M} > 1$ DEFINITIVAMENTE in n

---

$$\lim_n \frac{n!}{a^n} = +\infty \quad , \quad a > 1$$

$a = 5$

$$\frac{n!}{5^n} = \frac{n}{5}\,\frac{(n-1)}{5}\,\frac{(n-2)}{5} \cdots\cdots \frac{3}{5}\,\frac{2}{5}\,\frac{1}{5}$$

$\boxed{n \geq 7}$

$$= \frac{n}{5} \cdot \underbrace{\frac{(n-1)}{5}}_{\geq 1} \cdot \underbrace{\frac{(n-2)}{5}}_{\geq 1} \cdots \frac{4 \cdot 3 \cdot 2 \cdot 1}{5^4}$$

$$\geq \frac{n}{5} \cdot 1 \cdot 1 \cdots \frac{4!}{5^4} = n \cdot \frac{4!}{5^5}$$

$$\frac{n!}{5^n} \geq n\,\frac{4!}{5^5} > M \,, \quad n \geq 7$$

$$n > \frac{5^5}{4!} M$$

$$\nu_M = \max\left\{ \left[\frac{5^5}{4!} M \right] + 1 \,,\ 7 \right\}$$

$$\boxed{\frac{n!}{a^n} \geq C_a \cdot 2^n}$$

$$\lim_{n} \frac{n^{\alpha n}}{a^{\beta n}} = +\infty$$

$$\frac{n^{\alpha n}}{a^{\beta n}} = \left(\frac{n^\alpha}{a^\beta} \right)^n \overset{n \geq 1}{\geq} \frac{n^\alpha}{a^\beta} > M$$

definitivamente.

$$\frac{n^\alpha}{a^\beta} \longrightarrow +\infty \implies \frac{n^\alpha}{a^\beta} > M$$

$$\lim_{n} \frac{n^n}{n!} = +\infty$$

$$\boxed{n \geq 7}$$

$$\frac{n^n}{n!} = \frac{n}{n} \cdot \overset{\geq 1}{\frac{n}{n-1}} \cdot \overset{\geq 1}{\frac{n}{n-2}} \cdot \overset{\geq 1}{\frac{n}{n-3}} \cdots\cdots \frac{n}{3} \cdot \frac{n}{2} \cdot \frac{n}{1} \geq$$

$$\geq 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdots\cdots 1 \cdot 1 \cdot n > M$$

$$\forall n > \nu_M$$

$$\nu_M = [M] + 1$$

NOTA: Si dimostra con più attenzione che:

$$\frac{n^n}{n!} > 2^n, \ \forall n \geq 6$$

$\blacksquare$

$$\lim_{n} \frac{3^n - n^4}{n^n + n!} = \lim_{n} \frac{3^n\left(1 - \frac{n^4}{3^n}\right)}{n^n\left(1 + \frac{n!}{n^n}\right)} =$$

$$\frac{n^4}{3^n} \to 0, \quad \frac{n^n}{n!} \to +\infty$$

$$= \lim_{n} \overset{\nearrow 0}{\frac{3^n}{n^n}} \frac{(1 + o(1))}{(1 + o(1))} = 0 \cdot 1 = 0$$

$$\lim_{n} \frac{2^{2n} + n! + n^4}{5n^n + (\log_4(n))^7 - 6^n + 1} =$$

$$= \lim_{n} \frac{n!}{n^n} \frac{1 + \frac{4^n}{n!} + \frac{n^4}{n!}}{5 + \frac{(\log_4(n))^7}{n^n} - \frac{6^n}{n^n} + \frac{1}{n^n}} =$$

OSSERVIAMO CHE per $n \to +\infty$ si ha:

$$4^n / n! = o(1) \, ;$$

$$\frac{n^4}{n!} = \frac{n^4}{2^n} \cdot \frac{2^n}{n!} = o(1) \cdot o(1) = o(1) \, ;$$

$$\frac{(\log_4(n))^7}{n^n} = \frac{(\log_4(n))^7}{n} \, \frac{n}{2^n} \cdot \frac{2^n}{n^n} =$$

$$= o(1) \cdot o(1) \cdot o(1) = o(1)$$

Quindi:

$$\dots = \lim_{n} \frac{n!}{n^n} \, \frac{1 + o(1)}{5 + o(1)} = 0 \cdot \frac{1}{5} = 0$$

PROPRIETA' IMPORTANTI

$a_n = o(b_n)$ SE

$a_n \longrightarrow +\infty$, $b_n \longrightarrow +\infty$ e $\dfrac{a_n}{b_n} \xrightarrow[n \to +\infty]{} 0$,

$a_n = o(b_n)$, $b_n = o(c_n)$

$\Longrightarrow \dfrac{a_n}{c_n} = \dfrac{a_n}{b_n} \cdot \dfrac{b_n}{c_n} \longrightarrow 0 \Longrightarrow a_n = o(c_n)$

$a_n = \log_2(n)$, $b_n = n$, $c_n = 2^n$

$$\frac{(\log_a(n))^{10}}{2^n} \qquad (\log_a(n))^{10} = o(n)$$

$$n = o(2^n)$$

$$\Longrightarrow (\log_a(n))^{10} = o(2^n)$$

TEOREMI DI CONFRONTO PER LIMITI DI SUCCESSIONI

TEOREMA [DI PERMANENZA DEL SEGNO]

Sia a_n una SUCCESSIONE REGOLARE

SE $\lim_{n} a_n = a > 0$ ALLORA

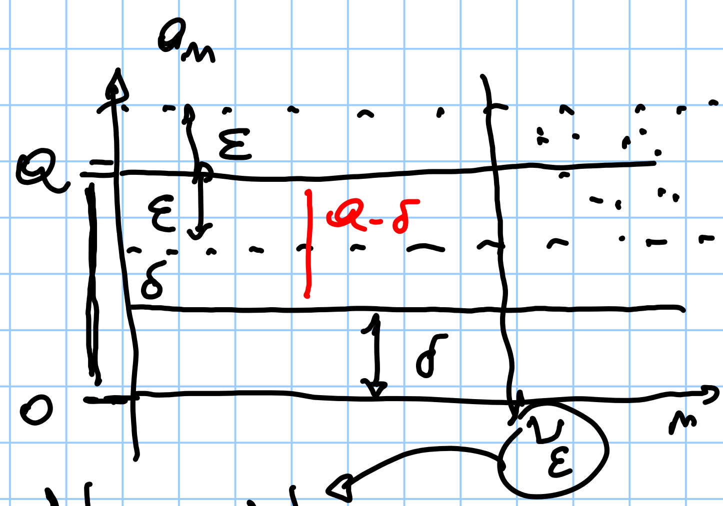
$\exists\, \delta > 0 : \quad a_n \geq \delta$
DEFINITIVAMENTE in n.

IN PARTICOLARE

$\forall\, \delta \in (0, a)\,,\ \exists\, \nu_\delta \in \mathbb{N}:$

$$a_n \geq \delta\,, \quad \forall\, n > \nu_\delta$$

DIM.



$\forall\, \varepsilon > 0\ \exists\, \nu_\varepsilon :$

$$|a_n - a| < \varepsilon, \forall\, n > \nu_\varepsilon$$

Sia $\varepsilon_\delta : 0 < \varepsilon_\delta < \underline{a - \delta}$ $\qquad \nu_\delta := \nu_{\varepsilon_\delta}$

$$a_n - a > -\varepsilon_\delta \qquad \forall\, n > \nu_\delta$$

$$a_n > a - \varepsilon_\delta > \delta \qquad \forall\, n > \nu_\delta$$

$\boxtimes$

$a_n \longrightarrow a < 0 \implies \exists\, \delta > 0 : a_n \leq -\delta$
DEFINITIVAMENTE

COROLLARIO

Se $a_n \geq 0$ definitivamente e $a_n \to a$,

ALLORA $a \geq 0$

DIMOSTRAZIONE

PER ASSURDO.

SUPPONIAMO $a < 0$.

Dalla PERMANENZA DEL SEGNO $a_n \leq -\delta$, $\delta > 0$ definitivamente.

Ma $a_n \geq 0$ definitivamente

ASSURDO $\boxtimes$

IMPORTANTE

Se $a_n \to 0$

il risultato non vale

- $a_n = \frac{1}{n}$, $a_n \to 0$

$a_n > 0 \quad \forall n \in \mathbb{N}$

ma non è VERO CHE

$a_n \geq \delta > 0$ definitivamente

- $a_n = \frac{(-1)^n}{n}$, $a_n \to 0$

il segno di a_n oscilla $a_{2k} > 0$

$a_{2k-1} < 0$