$$a_n = \frac{1}{n^2+1} + 7$$

$$\lim_{n \to +\infty} a_n = 7$$

$$(L) \quad \forall \varepsilon > 0 \ \exists \nu_\varepsilon \in \mathbb{N} : \ |a_n - 7| < \varepsilon, \ \forall n > \nu_\varepsilon$$

$$\nu_\varepsilon := \begin{cases} 1, & \varepsilon > 1 \\ \left[\sqrt{\frac{1}{\varepsilon} - 1}\right] + 1, & \varepsilon \leq 1 \end{cases}$$

$$(L1) \quad \forall \varepsilon > 0 \ \exists \nu_\varepsilon^{(1)} \in \mathbb{N} : \ |a_n - 7| < 4\varepsilon, \ \forall n > \nu_\varepsilon^{(1)}$$

$$|a_n - 7| < 4\varepsilon \iff \left| \frac{1}{n^2+1} - \cancel{7} + \cancel{7} \right| < 4\varepsilon$$

$$\cdots\cdots$$

$$n^2 > \frac{1}{4\varepsilon} - 1 \qquad \nu_\varepsilon^{(1)} = \begin{cases} 1, & 4\varepsilon > 1 \\ \left[\sqrt{\frac{1}{4\varepsilon} - 1}\right] + 1, & 4\varepsilon \leq 1 \end{cases}$$

$$\nu_\varepsilon^{(1)} = \nu_{4\varepsilon} \; ; \; \nu_\varepsilon = \left[\sqrt{\frac{1}{\varepsilon} - 1}\right] + 1$$

$$\varepsilon \longrightarrow 4\varepsilon$$

$$\nu_\varepsilon = \nu_{\varepsilon/4}^{(1)}$$

$$(L)_{\varepsilon} \Longleftrightarrow (L1)_{4\varepsilon}$$

$$\forall \varepsilon > 0 \; \exists \nu_\varepsilon \in \mathbb{N} : |a_n - L| < \varepsilon, \forall n > \nu_\varepsilon$$

$$\Updownarrow$$

$$\boxed{K > 0}$$

$$\forall \varepsilon > 0 \; \exists \nu_\varepsilon^{(1)} \in \mathbb{N} : |a_n - L| < K\varepsilon, \forall n > \nu_\varepsilon^{(1)}$$

$$\nu_\varepsilon \longrightarrow \nu_\varepsilon^{(1)} = \nu_{K\varepsilon}$$

$$\nu_\varepsilon^{(1)} \longrightarrow \nu_\varepsilon = \nu_{\varepsilon/K}^{(1)}$$

DEFINIZIONE

$$\lim_{n \to +\infty} a_n = +\infty \quad (\underline{-\infty})$$

$\forall M > 0 \ \exists \nu_M \in \mathbb{N} : a_n > M \quad (\underline{a_n < -M})$

$\forall n > \nu_M$

OVVERO : $\forall M > 0$

$a_n \in (M, +\infty)$ DEFINITIVAMENTE in n

$a_n \in (-\infty, -M)$ . . . .

$$\lim_{n \to +\infty} (-3^n) = -\infty$$

$\forall M > 0 \ \exists \nu_M \in \mathbb{N} : -3^n < -M$

$\forall n > \nu_M$

Ex

$$\lim_{n \to +\infty} -\log_3 n = -\infty$$

$$\forall M > 0 \ \exists \nu_M : -\log_3 n < -M \quad \forall n > \nu_M$$

$$-\log_3(n) < -M \iff$$

$$\log_3(n) > M \iff (3^x \nearrow)$$

$$3^{\log_3(n)} > 3^M \iff$$

$$n > 3^M.$$

$$\nu_M = [3^M] + 1$$

$$\lim_{n \to +\infty} a^{\beta n^{\gamma}} = +\infty$$

$$a > 1, \ \beta > 0, \ \gamma > 0.$$

$$\nu_M = \begin{cases} 1, & M \in (0,1] \\ \left[\sqrt[\gamma]{\log_a \sqrt[\beta]{M}}\right] + 1, & M > 1 \end{cases}$$

$$\boxed{m^\gamma > \log_a \sqrt[\beta]{M}}$$

$$\lim_{m \to +\infty} a_m = L, \; L \in \mathbb{R}$$

SUCCESSIONI CONVERGENTI

$$\lim_{m \to +\infty} a_m = +\infty \; (-\infty)$$

SUCCESSIONI DIVERGENTI

IN ENTRAMBE I CASI:

SUCCESSIONI REGOLARI,

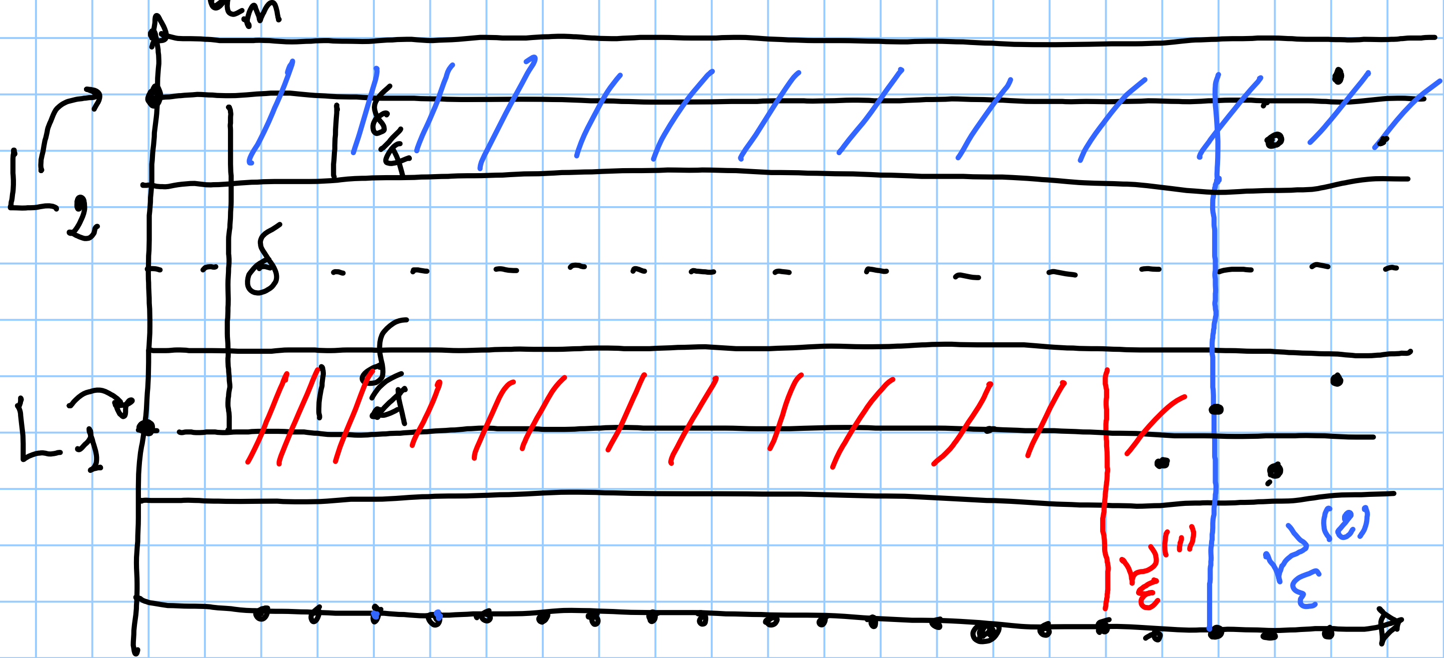
OVVERO, QUELLE IL CUI LIMITE ESISTE FINITO (L) O INFINITO ($\pm\infty$)

TEOREMA Sia $a_n \to L \in \overline{\mathbb{R}}$ $n \to +\infty$. ALLORA il LIMITE L è UNICO

DIMOSTRAZIONE. DIMOSTRIAMO CHE SE $a_n \to L_1$ e $a_n \to L_2$, $n \to +\infty$ $L_i \in \mathbb{R}$, $i=1,2$, ALLORA $L_1 = L_2$.

PER ASSURDO $L_1 \neq L_2$ (e $L_1 \in \mathbb{R}, L_2 \in \mathbb{R}$)

S.P.D.G. $L_2 - L_1 = \delta > 0$



$\forall \varepsilon > 0$ $\quad a_n \in (L_2 - \varepsilon, L_2 + \varepsilon)$ definitivamente in n.

$\boxed{\varepsilon = \delta/4}$ $\quad a_n \in (L_1 - \varepsilon, L_1 + \varepsilon) \quad = \quad =$

$\exists\, \nu^{(1)} \in \mathbb{N} : |a_n - L_1| < \delta/4 \quad \forall n > \nu^{(1)}$

$\exists\, \nu^{(2)} \in \mathbb{N} : |a_n - L_2| < \delta/4 \quad \forall n > \nu^{(2)}$

$\forall n > \max\{\nu^{(1)}, \nu^{(2)}\} \Longrightarrow$

$|a_n - L_1| < \delta/4 \quad e \quad |a_n - L_2| < \dfrac{\delta}{4}$

$$\delta = |L_2 - L_1| = |L_2 - a_n + a_n - L_1| =$$

$$= |\underbrace{(L_2 - a_n)}_{X} + \underbrace{(a_n - L_1)}_{Y}| \leq |L_2 - a_n| + |a_n - L_1|$$

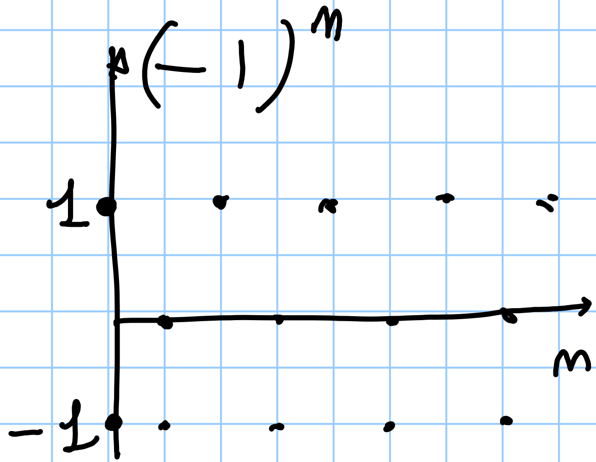
$$= |a_n - L_2| + |a_n - L_1| < \frac{\delta}{4} + \frac{\delta}{4} = \frac{\delta}{2}$$

$\delta < \dfrac{\delta}{2}$ $\qquad 1 < \dfrac{1}{2}$ ASSURDO

$a_n = 5 \quad , n \in \mathbb{N}$

$$\lim_{n \to +\infty} a_n = 5$$

$a_n = (-1)^n \quad , n \in \mathbb{N}$



DIMOSTRIAMO CHE NON ESISTE $\lim_{n \to +\infty} a_n$

PER ASSURDO $\exists\, L \in \mathbb{R}$:

$$\lim_{n \to +\infty} a_n = L$$

$$\forall \varepsilon > 0 \ \exists \nu_\varepsilon \in \mathbb{N} : |a_n - L| < \varepsilon, \forall n > \nu_\varepsilon$$

$$|(-1)^n - L| < \varepsilon \quad \forall n > \nu_\varepsilon$$

$$n > \nu_\varepsilon \quad n = 2k \quad , n \in 2\mathbb{N}$$

$$|1 - L| < \varepsilon \quad \forall k \in \mathbb{N}, \; 2k > \nu_\varepsilon$$

$$\Downarrow$$

$$|1 - L| < \varepsilon \quad \forall \varepsilon > 0 \Longrightarrow L = 1$$

$$n > \nu_\varepsilon \quad n = 2k-1 \quad , n \in 2\mathbb{N}-1$$

$$|(-1) - L| < \varepsilon \quad \forall k \in \mathbb{N},$$

$$2k-1 > \nu_\varepsilon$$

$$|(-1) - L| < \varepsilon \quad \forall \varepsilon > 0 \Longrightarrow L = -1$$

ASSURDO

LE SUCCESSIONI IL CUI LIMITE NON ESISTE SI DICONO IRREGOLARI

TEOREMA Sia $a_n \longrightarrow L$, $n \to +\infty$.

SE $L \in \mathbb{R}$, ALLORA

a_n è LIMITATA.

DIM.

$L \in \mathbb{R}$, $\varepsilon = 1$, $\exists\, \nu_1 \in \mathbb{N}$:

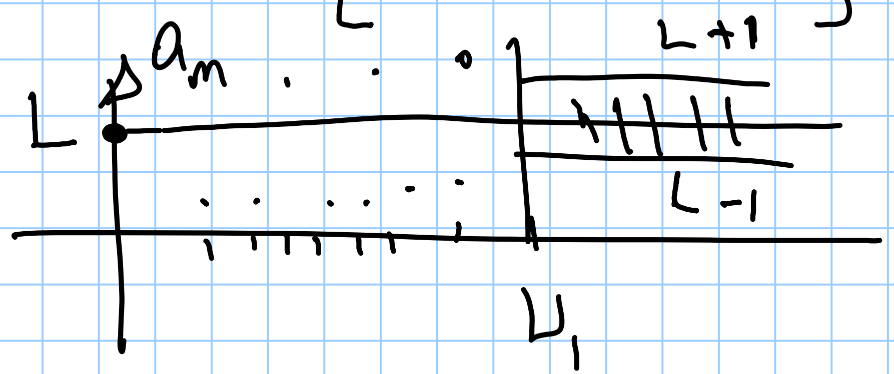
$$|a_n - L| < 1 \quad \forall n > \nu_1$$

$$|a_n| = |a_n - L + L| \le |a_n - L| + |L|$$

$$< 1 + |L| \quad \forall n > \nu_1$$

$$M := \max_{n \in \{1, 2, \dots, \nu_1\}} |a_n|$$

$$|a_n| \le \max\{M,\ 1 + |L|\}$$



∎

TEOREMA

Siano a_n e b_n due successioni REGOLARI. Se la somma dei limiti non è una delle FORME INDETERMINATE $(\pm\infty)+(\mp\infty)$, $(\pm\infty)-(\pm\infty)$, ALLORA

Si ha: $$\lim_{n\to+\infty}(a_n \pm b_n) = \lim_{n\to+\infty} a_n \pm \lim_{n\to+\infty} b_n$$

Se il prodotto dei limiti NON E' UNA delle FORME INDETERMINATE $0\cdot(\pm\infty)$, $(\pm\infty)\cdot 0$, ALLORA

Si ha: $$\lim_{n\to+\infty}(a_n\cdot b_n) = \left(\lim_{n\to+\infty} a_n\right)\cdot\left(\lim_{n\to+\infty} b_n\right)$$

Se il RAPPORTO dei limiti NON E' UNA DELLE F.I. $\frac{\pm\infty}{\pm\infty}$, $\frac{\pm\infty}{\mp\infty}$ e SE $\lim_{n\to+\infty} b_n \neq 0$ ALLORA

SI HA $\lim\limits_{n \to +\infty} \left(\frac{a_n}{b_n} \right) = \frac{\lim\limits_{n \to +\infty} a_n}{\lim\limits_{n \to +\infty} b_n}$

$\lim\limits_{n} a_n = a \in \mathbb{R}$, $\lim\limits_{n} b_n = b \in \mathbb{R}$.

$$\lim_{n} (a_n + b_n) = \lim_{n} a_n + \lim_{n} b_n = a + b$$

IL LIMITE DELLA SOMMA
E' UGUALE
ALLA SOMMA DEI LIMITI
$(a+b)$

$$\lim_{n} (a_n + b_n) = a + b$$

$\forall \varepsilon > 0 \quad \exists \, \nu_\varepsilon^{(1)} : |a_n - a| < \varepsilon \; \forall n > \nu_\varepsilon^{(1)}$

$\exists \, \nu_\varepsilon^{(2)} : |b_n - b| < \varepsilon \; \forall n > \nu_\varepsilon^{(2)}$

$\exists \quad \nu_\varepsilon^{(1)} \;\mathrm{e}\; \nu_\varepsilon^{(2)}$ (PER IPOTESI).

$$|(a_n+b_n)-(a+b)| = |\underbrace{(a_n-a)}_{x}+\underbrace{(b_n-b)}_{y}|$$

$$\le |a_n-a|+|b_n-b| < \varepsilon+\varepsilon = 2\varepsilon.$$

$$n > \max\{\nu_\varepsilon^{(1)}, \nu_\varepsilon^{(2)}\} =: \nu_\varepsilon$$

$$\forall \varepsilon>0 \ \exists \nu_\varepsilon : |(a_n+b_n)-(a+b)| < 2\varepsilon \quad \forall n > \nu_\varepsilon$$

$$\Longleftrightarrow \quad \lim_n (a_n+b_n) = a+b$$

$\not\!\!\emptyset$

$$\lim_n a_n b_n = a\cdot b \quad , \quad \lim_n a_n = a \in \mathbb{R}$$

$$\lim_n b_n = b \in \mathbb{R}$$

$$\nu_\varepsilon^{(1)}, \ a \quad , \quad \nu_\varepsilon^{(2)}, \ b.$$

$$|a_n b_n - ab| = |a_n b_n - a b_n + a b_n - ab| =$$

$$= |\underbrace{b_n(a_n-a)}_{x} + \underbrace{a(b_n-b)}_{y}| \le$$

$$|b_n(a_n-a)| + |a(b_n-b)| \leq$$

$$|b_n|\,|a_n-a| + |a|\,|b_n-b| \leq$$

b_n CONVERGENTE $\Longrightarrow$ b_n LIMITATA $\Longrightarrow$

$$\exists\, M>0 : \ |b_n| \leq M \quad \forall n \in \mathbb{N}$$

$$M\,|a_n-a| + |a|\,|b_n-b| <$$

$$n > \max\{\nu_\varepsilon^{(1)}, \nu_\varepsilon^{(2)}\} =: \nu_\varepsilon$$

$$< M\,\varepsilon + |a|\,\varepsilon = (M+|a|)\,\varepsilon$$

$$\forall \varepsilon>0 \ \exists\, \nu_\varepsilon : |a_n \cdot b_n - ab| < (M+|a|)\,\varepsilon$$

$$\forall n > \nu_\varepsilon$$

$$\Longleftrightarrow \quad \lim_n a_n b_n = a \cdot b \, .$$

$a_n \longrightarrow +\infty$, $b_n \longrightarrow b \in \mathbb{R}$ Ex

. . . . .

$\boxed{\varepsilon}$

OSSERVAZIONE

Questo teorema mostra la buona posizione delle regole di somma e prodotto di $x \in \mathbb{R}$ con $\pm\infty$.

Per esempio, se $x \in \mathbb{R}$ e

$$a_n \xrightarrow[n \to +\infty]{} x, \quad b_n \xrightarrow[n \to +\infty]{} +\infty$$

$$\Longrightarrow \quad a_n + b_n \xrightarrow[n \to +\infty]{} +\infty$$

$[+\infty + x = +\infty]$

$\lim_n b_n = 0$, $b_n \neq 0 \ \forall n$, $\lim_n \frac{a_n}{b_n}$ e SE

NON HO FORME INDETERMINATE

"$\frac{0}{0}$" ALLORA $\lim_n \left|\frac{a_n}{b_n}\right| = +\infty$

$a_n = 1$

$b_n = \frac{1}{n}$, $\frac{1}{b_n} = n \longrightarrow +\infty$

$b_n = -\frac{1}{n}$, $\frac{1}{b_n} = -n \longrightarrow -\infty$

$b_n = \frac{(-1)^n}{n}$ $\frac{1}{b_n} = (-1)^n n$ IRREGOLARE

$\lim_n n^2 + 5 = +\infty + 5 = +\infty$

$\lim_n \frac{3 + \frac{1}{n^4}}{4 + \frac{1}{(\log_e n)^7}} = \frac{3 + 0}{4 + 0} = \frac{3}{4}$

FORMA INDETERMINATA

$a_n = n^2 \quad , \quad b_n = n$

$a_n \to +\infty \quad , \quad b_n \to +\infty \quad , \quad n \to +\infty$

$$\lim_n \frac{a_n}{b_n} = \frac{+\infty}{+\infty} \ ??$$

$$\frac{a_n}{b_n} = \frac{n^2}{n} = n \longrightarrow +\infty$$

$a_n = n \quad , \quad b_n = n^2$

$a_n \to +\infty \quad , \quad b_n \to +\infty \quad , \quad n \to +\infty$

$$\frac{a_n}{b_n} = \frac{n}{n^2} = \frac{1}{n} \longrightarrow = \frac{1}{+\infty} = 0$$

$a_n = n^2 + 1 \quad , \quad b_n = n^2$

$$\frac{a_n}{b_n} = \frac{n^2+1}{n^2} = 1 + \frac{1}{n^2} \longrightarrow 1 + \frac{1}{+\infty} =$$
$$= 1 + 0 = 1$$

$$\frac{\pm\infty}{+\infty}\ ,\quad a_n \to \pm\infty\ ,\quad b_n \to +\infty$$

$$\lim_n \frac{a_n}{b_n} = x \in \mathbb{R}$$

$$a_n = x n^2 + 1 \qquad \frac{x n^2 + 1}{n^2} = x + \underbrace{\frac{1}{n^2}}_{\to 0}$$

$$b_n = n^2$$

$$\lim_n \frac{a_n}{b_n} = x$$

$$+\infty - (+\infty)$$

$$n^2 - n \longrightarrow +\infty - (+\infty)$$

$$n - n^2 \longrightarrow +\infty - (+\infty)$$

$$\lim_n (n^2 - n) = \lim_n \underbrace{n^2}_{\downarrow} \underbrace{\left(1 - \frac{1}{n}\right)}_{} =$$

$$= +\infty \cdot 1 = +\infty$$

$$\lim_n (n - n^2) = \lim_n n^2\left(\frac{1}{n} - 1\right) = -\infty$$

$$a_n = n^2 + x \quad , \quad b_n = n^2$$

$$a_n \longrightarrow +\infty \, , \quad b_n \longrightarrow +\infty$$

$$\lim_n (a_n - b_n) = x$$