$L \in \mathbb{R}$, $\quad I = \left\{ x \in \mathbb{R} : |x - L| < \underset{(\varepsilon)}{1} \right\}$

$\varepsilon > 0$

Intervallo CENTRATO in L

di AMPIEZZA 2 (2ε)

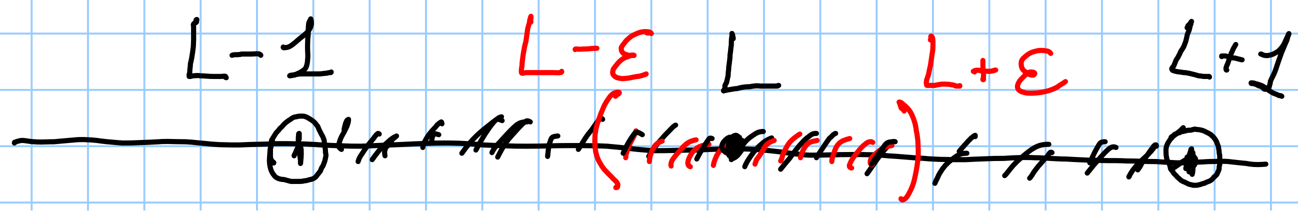
$$I = \left\{ x \in \mathbb{R} : x \in \left(L - \underset{\varepsilon}{1}, L + \underset{\varepsilon}{1} \right) \right\}$$

$$= \left\{ x \in \mathbb{R} : L - \underset{\varepsilon}{1} < x < L + \underset{\varepsilon}{1} \right\}$$

$$= \left\{ x \in \mathbb{R} : -\underset{\varepsilon}{1} < x - L < \underset{\varepsilon}{1} \right\}$$

DIMOSTRAZIONE

$$|x - L| < 1 \iff$$



$$\begin{cases} x - L < 1 \\ x \geq L \end{cases} \cup \begin{cases} -x + L < 1 \\ x < L \end{cases}$$

$$\begin{cases} x < L + 1 \\ x \geq L \end{cases} \cup \begin{cases} x > L - 1 \\ x < L \end{cases}$$

$$\Longleftrightarrow \left\{ x \in \mathbb{R} : L - 1 < x < L + 1 \right\}$$

$$\Longleftrightarrow \left\{ x \in \mathbb{R} : -1 < x - L < 1 \right\}$$

$$\Longleftrightarrow \left\{ x \in \mathbb{R} : x \in (L - 1, L + 1) \right\}$$

DEFINIZIONE

Sia $\mathcal{P}_n$ una PROPRIETÀ, $n \in \mathbb{N}$

Si dice che $\mathcal{P}_n$ VALE DEFINITIVAMENTE in n SE

$\exists\, \nu \in \mathbb{N} :$ $\mathcal{P}_n$ È VERA $\forall n > \nu$

ESEMPIO

ν = " ENNE MINUSCOLA GRECA " $(\to$ Ni$)$

$\mathcal{P}_n$ = " $b_n \leq a_n$ "

$$b_n = -3^n + (100 - n) \quad ,$$
$$a_n = -3^n \quad ,$$

$$\cancel{-3^n} + (100-n) \le \cancel{-3^n} \iff$$

$$n \geq 100$$

$n = 1$ E' FALSA

$n = 2$ E' FALSA

$\vdots$ ....

$n = 99$ E' FALSA

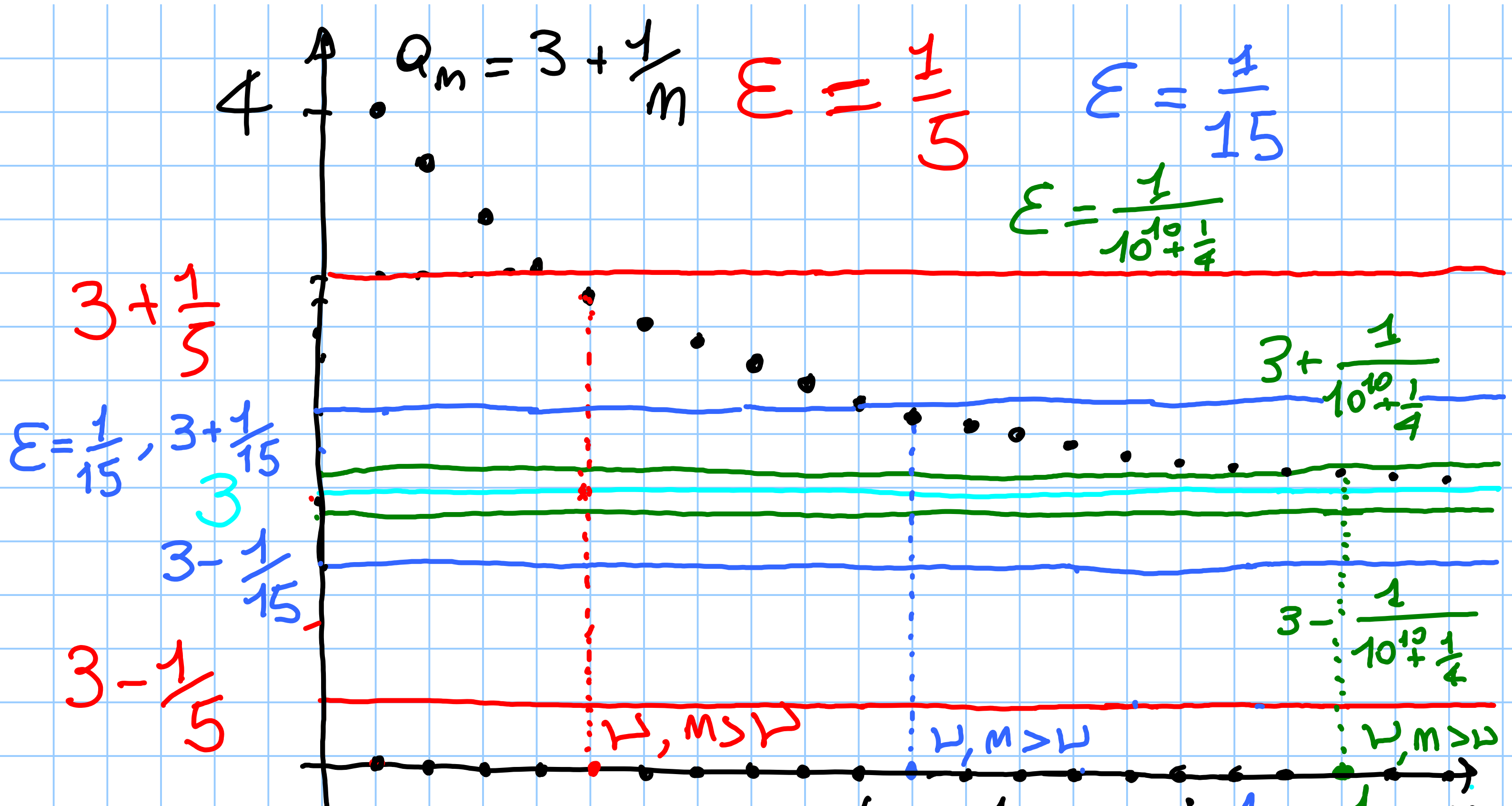
$\boxed{\nu = 99} \Longrightarrow$ E' VERA $\forall n > \nu$

$n \geq 100 \quad \forall n > 99$

$\nu = 1000 \Longrightarrow$ E' VERA $\forall n > \nu$

P_n E' VERA DEFINITIVAMENTE in n

$$a_n = 3 + 1/n \quad , n \in \mathbb{N}$$



È VERO CHE $a_n \in \left(3-\frac{1}{5}, 3+\frac{1}{5}\right)$ $\frac{1}{15}$, $\frac{1}{10^{10}+\frac{1}{4}}$

DEFINITIVAMENTE in n?

È VERO CHE $\exists \nu \in \mathbb{N}: a_n \in \left(3-\frac{1}{5}, 3+\frac{1}{5}\right), \forall n>\nu$

" " $\exists \nu \in \mathbb{N}: 3-\frac{1}{5} < a_n < 3+\frac{1}{5}, \forall n>\nu$

" " $\exists \nu \in \mathbb{N}: -\frac{1}{5} < a_n - 3 < \frac{1}{5}, \forall n>\nu$

" " $\exists \nu \in \mathbb{N}: |a_n - 3| < \frac{1}{5}, \forall n>\nu$

$\exists \nu \in \mathbb{N}: |a_n - 3| < \frac{1}{15}, \forall n>\nu$

$$|a_n - 3| < \frac{1}{5} \Longleftrightarrow \left|\cancel{3} + \frac{1}{n} - \cancel{3}\right| < \frac{1}{5}$$

$\frac{1}{15}, \frac{1}{10^{10}+\frac{1}{4}}$ $\qquad \frac{1}{15}$

$$\Longleftrightarrow \left|\frac{1}{n}\right| < \frac{1}{5} \quad \frac{1}{15} \qquad -\frac{1}{5} < \frac{1}{n} < \frac{1}{5} \quad \frac{1}{15}$$

$-\frac{1}{15}$

$-\frac{1}{10^{10}+\frac{1}{4}}, -\frac{1}{15}$

$$\begin{cases} -\frac{1}{5} < \frac{1}{n} & \text{È VERA } \forall n \in \mathbb{N} \\ \frac{1}{n} < \frac{1}{5} \quad \frac{1}{15} \quad \frac{1}{10^{10}+\frac{1}{4}} \end{cases}$$

È VERA $\forall n \in \mathbb{N}$

È VERA $\forall n \in \mathbb{N}$

$n > 15$

$$\Longleftrightarrow \frac{1}{n} < \frac{1}{5} \Longleftrightarrow n > 5$$

$\boxed{\nu = 5}$ $\qquad$ È VERA $\forall n > \nu_{\frac{1}{5}} = 5$

$\frac{1}{15}$ $\qquad \boxed{\nu = 15}$

È VERA $\forall n > \nu_{\frac{1}{15}} = 15$

È VERO CHE $\exists \nu \in \mathbb{N}$: $|a_n - 3| < \frac{1}{10^{10}+\frac{1}{4}} \quad \forall n > \nu$?

$$\frac{1}{m} < \frac{1}{10^{10}+\frac{1}{4}} \qquad m > 10^{10} + \frac{1}{4}$$

$$\nu = \left[10^{10} + \frac{1}{4}\right] = 10^{10}$$

È VERA $\forall m > \nu_{\frac{1}{10^{10}+\frac{1}{4}}} = 10^{10}$

OSSERVAZIONE IMPORTANTISSIMA

$$\varepsilon = \frac{1}{5} \Longrightarrow \nu = \nu_{\frac{1}{5}} = 5$$

$$\varepsilon = \frac{1}{15} \Longrightarrow \nu = \nu_{\frac{1}{15}} = 15$$

$$\varepsilon = \frac{1}{10^{10}+\frac{1}{4}} \Longrightarrow \nu = \nu_{\frac{1}{10^{10}+\frac{1}{4}}} = 10^{10}$$

ν DIPENDE DA $\varepsilon \rightarrow \nu = \nu_\varepsilon$!

ν_ε È UNA FUNZIONE di ε !

È VERO CHE

$$\forall \varepsilon > 0, \exists \nu_\varepsilon \in \mathbb{N} : |a_n - 3| < \varepsilon, \forall n > \nu_\varepsilon ?$$

$$\left| 3 + \frac{1}{n} - 3 \right| < \varepsilon \iff$$

$$\begin{cases} -\varepsilon < \frac{1}{n} & \text{VERA } \forall n \in \mathbb{N} \\ \frac{1}{n} < \varepsilon \end{cases}$$

$$\iff \frac{1}{n} < \varepsilon \iff n > \frac{1}{\varepsilon}$$

$$\nu_\varepsilon = \left[\frac{1}{\varepsilon} \right] + 1$$

È VERA $\forall n > \nu_\varepsilon$

DEFINIZIONE Sia a_n una SUCCESSIONE e $L \in \mathbb{R}$.

SI DICE CHE "a_n TENDE a L per n CHE TENDE A $+\infty$" O CHE "IL LIMITE DI a_n PER n CHE TENDE A $+\infty$ E' L"

$$\lim_{n \to +\infty} a_n = L \,,$$

$$a_n \longrightarrow L \,, \quad n \longrightarrow +\infty \,,$$ SE

$$\forall \varepsilon > 0 \ \exists \nu_\varepsilon \in \mathbb{N} : |a_n - L| < \varepsilon, \forall n > \nu_\varepsilon$$

EQUIVALENTEMENTE:

$\forall \varepsilon > 0, a_n \in (L-\varepsilon, L+\varepsilon)$ DEFINITIVAMENTE in n

$$a_n = 3 + \frac{1}{n} \,, \quad \lim_{n \to +\infty} a_n = 3$$

SUCCESSIONI INFINITESIME

$$\lim_{n \to +\infty} a_n = 0$$

$$a_n = \frac{1}{\left(\log_a (n^\gamma)\right)^\beta}, \quad a>1, \quad \gamma > 0, \quad \beta > 0$$

$$a_n = \frac{1}{n^\alpha}, \quad \alpha > 0$$

$$a_n = \frac{1}{a^{\beta n^\gamma}}, \quad a>1, \quad \beta>0, \quad \gamma>0.$$

$$a_n = \frac{1}{(n!)^\beta}, \quad \beta > 0$$

$$a_n = \frac{1}{n^{\beta n}}, \quad \beta > 0$$

$$\forall \varepsilon > 0, \exists \nu_\varepsilon \in \mathbb{N} : |a_n - L| < \varepsilon, \forall n > \nu_\varepsilon$$

$$\lim_{n \to +\infty} a_n = L$$

SUPPONIAMO di AVER TROVATO $\varepsilon_0 > 0$:

$$\forall \varepsilon \in (0, \varepsilon_0] \ \exists \nu_\varepsilon \in \mathbb{N} : |a_n - L| < \varepsilon \ \forall n > \nu_\varepsilon$$

Se $\varepsilon > \varepsilon_0$ PONIAMO $\nu_\varepsilon := \nu_{\varepsilon_0}$.

ALLORA SI HA

$$|a_n - L| < \varepsilon_0 < \varepsilon, \quad \forall n > \nu_\varepsilon \ (= \nu_{\varepsilon_0})$$

A $\quad \forall \varepsilon > 0 \ \exists \nu_\varepsilon \in \mathbb{N} : |a_n - L| < \varepsilon \ \forall n > \nu_\varepsilon$

$$\Longleftrightarrow$$

B

$$\exists \varepsilon_0 > 0 : \forall \varepsilon \in (0, \varepsilon_0], \exists \nu_\varepsilon : |a_n - L| < \varepsilon \ \forall n > \nu_\varepsilon$$

$\boxed{\Longrightarrow}$ $A \Longrightarrow B$ IMMEDIATO

$$\nu_\varepsilon^{(B)} := \nu_\varepsilon^{(A)}, \quad \varepsilon \in (0, \varepsilon_0] \quad \varepsilon_0 > 0.$$

$\boxed{\Longleftarrow}$ $A \Longleftarrow B$

$\varepsilon \in (0, \varepsilon_0]$ $\quad \exists\, \nu_\varepsilon^{(B)} : |a_n - L| < \varepsilon \quad \forall n > \nu_\varepsilon^{(B)}$

Se $\varepsilon > \varepsilon_0$ DEFINIAMO $\nu_\varepsilon^{(A)} := \nu_{\varepsilon_0}^{(B)}$. ALLORA

$$|a_n - L| < \varepsilon_0 < \varepsilon \quad \forall n > \nu_\varepsilon^{(A)} = \left(\nu_{\varepsilon_0}^{(B)}\right)$$

$$\Longrightarrow \quad \nu_\varepsilon^{(A)} = \begin{cases} \nu_\varepsilon^{(B)} & \text{se } \varepsilon \in (0, \varepsilon_0] \\ \nu_{\varepsilon_0}^{(B)} & \text{se } \varepsilon > \varepsilon_0 \end{cases},$$

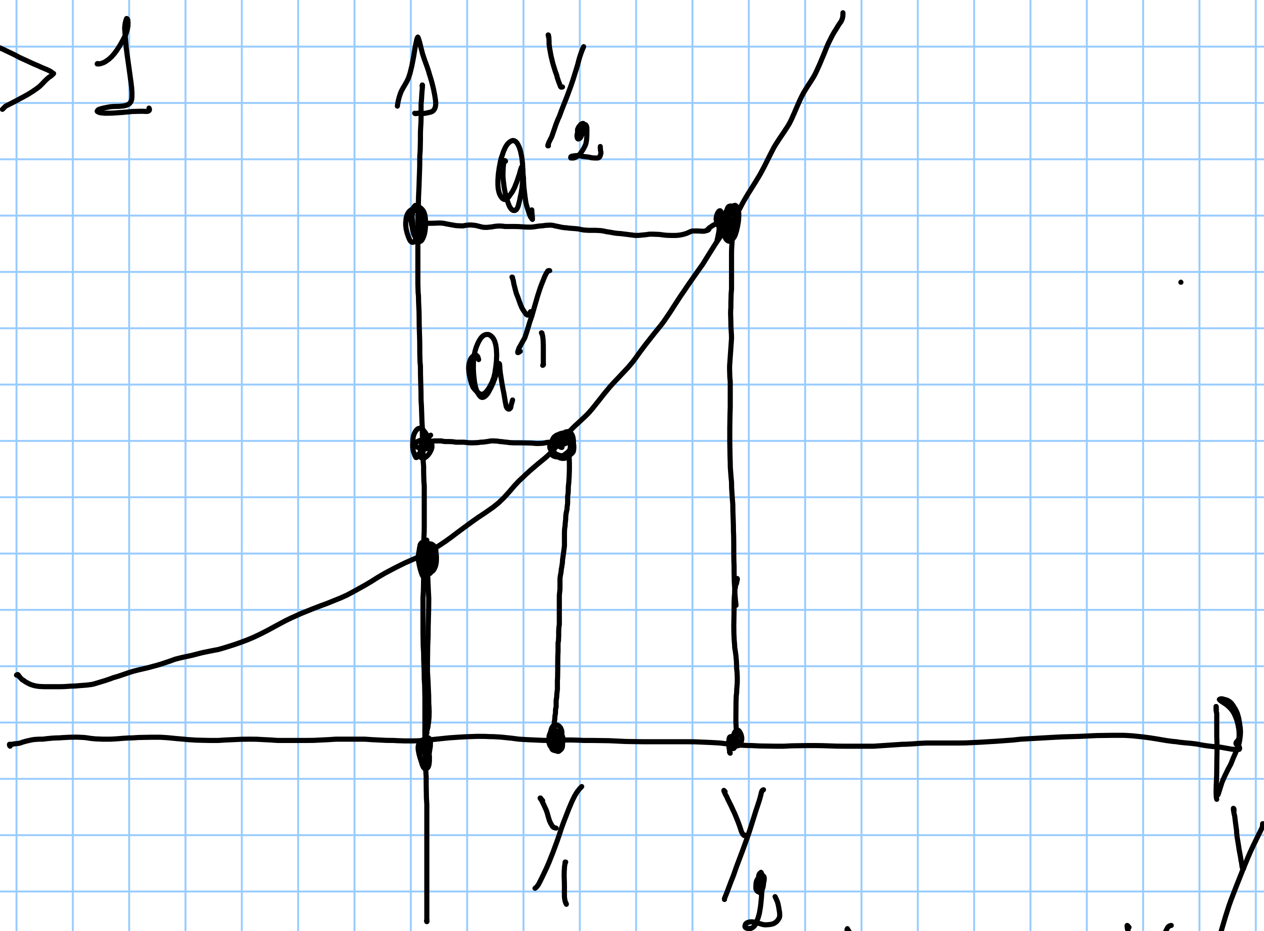
OSSERVAZIONE SULLE DISEQUAZIONI

Vogliamo risolvere

$$\log_2(x) > 3$$

Osserviamo che se $a > 1$ a^y è STRETTAMENTE CRESCENTE

$a > 1$



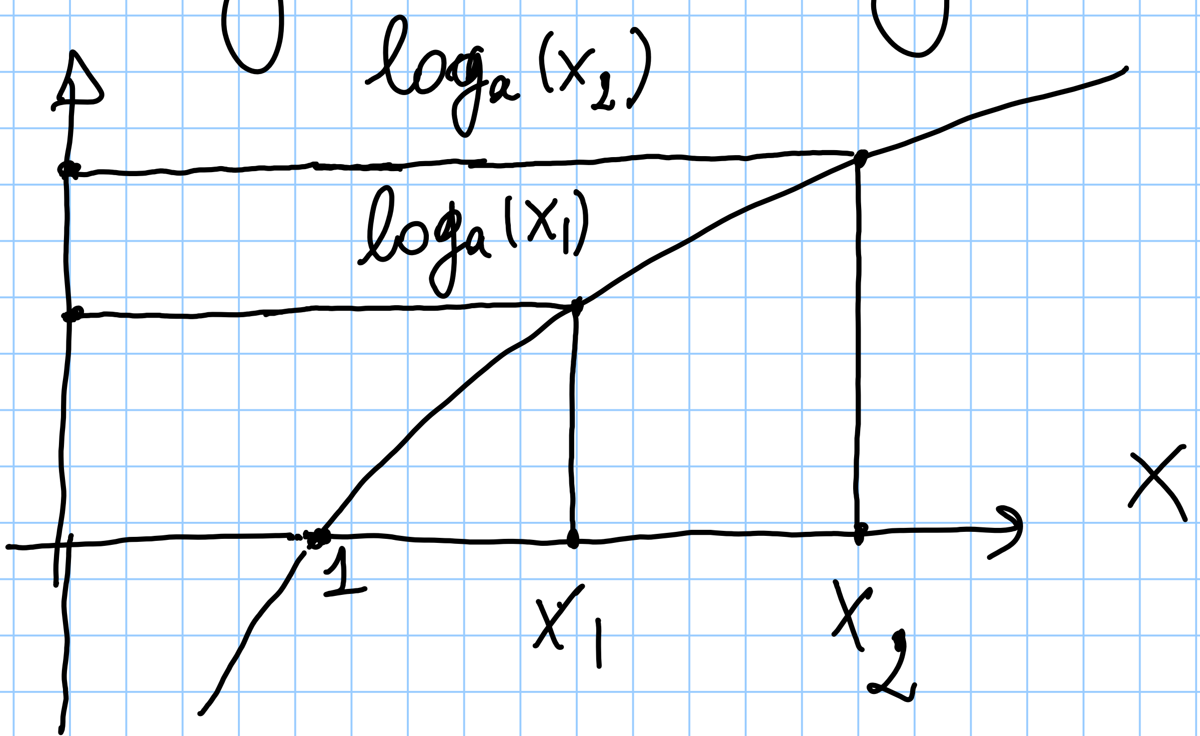
$$y_2 > y_1 \iff a^{y_2} > a^{y_1}$$

Quindi si ha $\left(y_2 = \log_2(x),\ y_1 = 3\right)$

$$\log_2(x) > 3 \iff x > 2^3$$

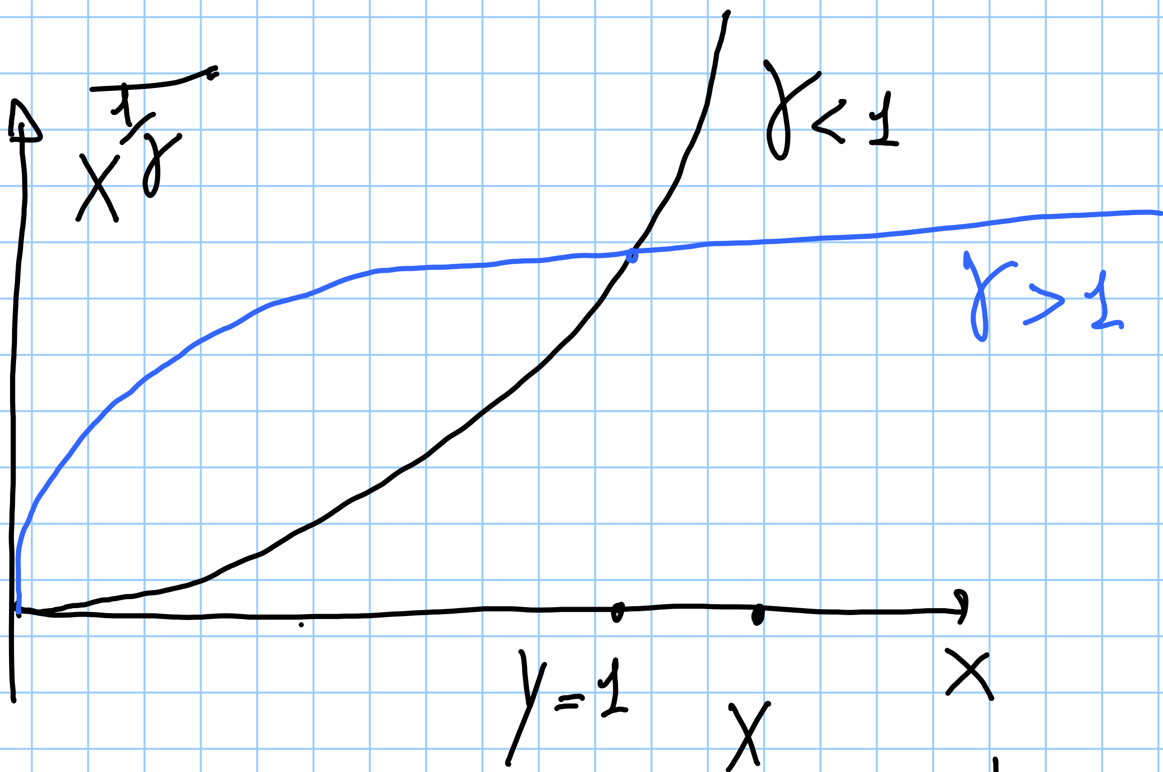
$a > 1$; $h(y) = \log_a(y)$ E' STRETTAMENTE CRESCENTE, quindi

$$x_2 > x_1 > 0 \iff \log_a(x_2) > \log_a(x_1)$$



$$\log_2(x) > 3 = \log_2(8) \iff x > 8$$

$\gamma > 0$



$$x > y \geq 0 \iff x^{\frac{1}{\gamma}} > y^{\frac{1}{\gamma}}$$

ESEMPIO $\quad n^{\gamma} > 1 \,, \; n \in \mathbb{N}$

$$\iff \quad n > 1^{1/\gamma} = 1$$

$$\sqrt{x^2 - 1} > x$$

$\gamma = \frac{1}{2}$

$$\begin{cases} x^2 - 1 \geq 0 \\ x \geq 0 \\ x^2 - 1 > x^2 \end{cases}$$

$\cup$

$$\begin{cases} x^2 - 1 \geq 0 \\ x < 0 \end{cases}$$

$\{x \leq -1\}$

$\gamma = 2$

$$n^2 > \frac{1}{\varepsilon} - 1 \quad , \; n \in \mathbb{N}, \; \varepsilon > 0$$

$$\begin{cases} \frac{1}{\varepsilon} - 1 \geq 0 \\ n > \sqrt{\frac{1}{\varepsilon} - 1} \end{cases} \cup$$

$$\begin{cases} \frac{1}{\varepsilon} - 1 < 0 \\ n \in \mathbb{N} \end{cases}$$

$$\left\{ n > \sqrt{\frac{1}{\varepsilon} - 1} \quad \text{se } \varepsilon \leq 1 \right\} \cup$$

$$\left\{ n \in \mathbb{N} \quad \text{Se } \varepsilon > 1 \right\}$$

$$a_n = \frac{1}{n^2+1} + 7$$

$$\lim_{n \to +\infty} a_n = 7$$

$$\forall \varepsilon>0 \quad \exists\, \nu_\varepsilon \in \mathbb{N} : |a_n - 7| < \varepsilon, \forall n > \nu_\varepsilon$$

$$\left| \frac{1}{n^2+1} + \cancel{7} - \cancel{7} \right| < \varepsilon$$

$$\left| \frac{1}{n^2+1} \right| < \varepsilon \qquad \begin{cases} -\varepsilon < \dfrac{1}{n^2+1} & \text{VERA } \forall n \in \mathbb{N} \\ \dfrac{1}{n^2+1} < \varepsilon \end{cases}$$

$$\Longleftrightarrow \quad \frac{1}{n^2+1} < \varepsilon \Longleftrightarrow n^2+1 > \frac{1}{\varepsilon}$$

$\varepsilon \le 1$

$$\boxed{n^2 > \frac{1}{\varepsilon} - 1} \Longleftrightarrow n > \sqrt{\frac{1}{\varepsilon} - 1}$$

Se $\varepsilon > 1 \Longrightarrow$ E' SEMPRE VERA

$\Longrightarrow$ Poniamo $\nu_\varepsilon = 1$ se $\varepsilon > 1$

$\forall \varepsilon > 1, |a_n - 7| < \varepsilon \quad \forall n > \nu_\varepsilon \ (=1)$.

Se $\varepsilon \leq 1 \Rightarrow \nu_\varepsilon := \left[\sqrt{\frac{1}{\varepsilon} - 1}\right] + 1$

$$\nu_\varepsilon := \begin{cases} 1 & , \ \varepsilon > 1 \\ \left[\sqrt{\frac{1}{\varepsilon} - 1}\right] + 1, & \varepsilon \in (0, 1] \end{cases}$$

ALTRO MODO

$\varepsilon \leq 1$

$$n^2 > \frac{1}{\varepsilon} - 1 \Leftarrow n > \sqrt{\frac{1}{\varepsilon} - 1}$$

E' SUFFICENTE TROVARE

ν_ε per $\varepsilon \in (0, \varepsilon_0]$ con

$\varepsilon_0 > 0$ FISSATO

Sia $\varepsilon_0 = 1$. Se $0 < \varepsilon \leq \varepsilon_0$

$$\nu_\varepsilon := \left[\sqrt{\frac{1}{\varepsilon} - 1}\right] + 1$$

IN QUESTO MODO EVITIAMO IL CASO $\varepsilon > 1$

$$\lim_{n \to +\infty} \frac{1}{\left(\log_a(n^\gamma)\right)^\beta} = 0 \quad , \quad a > 1, \quad \gamma > 0, \ \beta > 0.$$

$$\forall \varepsilon > 0 \ \exists \nu_\varepsilon \in \mathbb{N} : \left| \frac{1}{\left(\log_a(n^\gamma)\right)^\beta} \right| < \varepsilon \quad \forall n > \nu_\varepsilon$$

$$-\varepsilon < \frac{1}{\left(\log_a(n^\gamma)\right)^\beta} < \varepsilon \quad , \quad n \geq 2$$

$$\log_a(n^\gamma) > 0 \quad \forall \ n \geq 2$$

Infatti $\log_a(n^\gamma) > 0 \iff$

$$n^\gamma > a^0 = 1 \iff n > 1^{1/\gamma} = 1$$

Perché si ha

$f(x) = x^{1/\gamma}$ STRETTAMENTE
$\gamma > 0$ CRESCENTE in $[0, +\infty)$

IN PARTICOLARE

$$-\varepsilon < \frac{1}{\left(\log_a(n^\gamma)\right)^\beta} \quad \forall n \geq 2$$

$$\frac{1}{\left(\log_a(n^\gamma)\right)^\beta} < \varepsilon \Longleftrightarrow \left(\log_a(n^\gamma)\right)^\beta > \frac{1}{\varepsilon}$$

$h(y) = y^{1/\beta}$, $y \in [0, +\infty)$, $\beta > 0$
E' STRETTAMENTE $\nearrow$

$$\left(\log_a(n^\gamma)\right)^\beta > \frac{1}{\varepsilon} \Longleftrightarrow \log_a(n^\gamma) > \frac{1}{\sqrt[\beta]{\varepsilon}}$$

$g(y) = a^y$, $y \in \mathbb{R}$ E' STRETTAMENTE $\nearrow$

$$\Longleftrightarrow \quad n^{\gamma} > a^{1/\sqrt[\beta]{\varepsilon}} \Longleftrightarrow$$

$f(x) = x^{1/\gamma}$, $x \in [0,+\infty)$ E' STRETTAMENTE $\nearrow$

$$n > \left(a^{1/\sqrt[\beta]{\varepsilon}}\right)^{1/\gamma} = a^{\frac{1}{\gamma \sqrt[\beta]{\varepsilon}}}$$

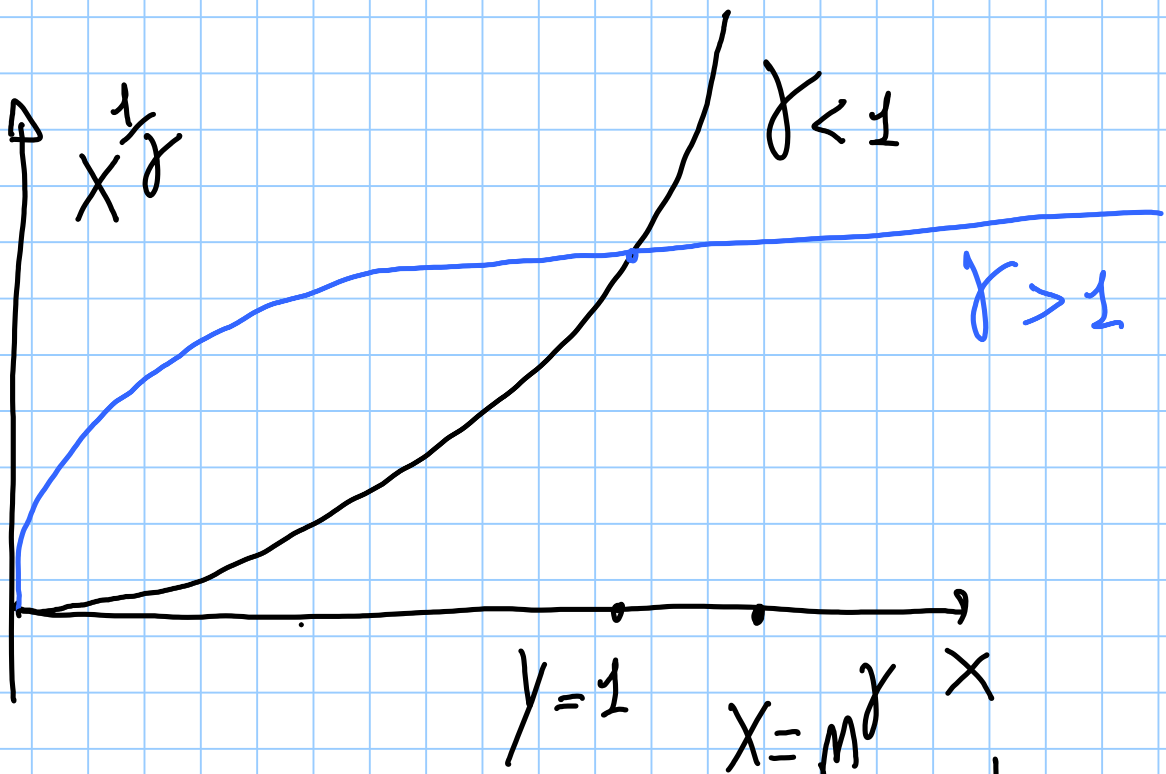
$$\nu_{\varepsilon} := \left[a^{\frac{1}{\gamma \sqrt[\beta]{\varepsilon}}} \right] + 1$$

$\blacksquare$

NOTA: $\left(\log_2(n^2)\right) \neq \left(\log_2(n)\right)^2$

$n = 2$; $\log_2(2^2) = 2$; $\left(\log_2(2)\right)^2 = 1$

$$\boxed{m^{\gamma} > 1}$$



$$x > y \iff x^{\frac{1}{\gamma}} > y^{\frac{1}{\gamma}}$$

$$\left(x = m^{\gamma} \,,\, y = 1 \right)$$

$$\iff (m^{\gamma})^{1/\gamma} > 1^{1/\gamma} \iff m > 1$$